

A TOUR OF STABLE REDUCTION WITH APPLICATIONS

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ABSTRACT. The Deligne–Mumford stable reduction theorem asserts that for a family of stable curves over the punctured disk, after a finite base change, the family can be completed in a unique way to a family of stable curves over the disk. In this survey, we discuss stable reduction theorems in a number of different contexts. This includes a review of recent results on abelian varieties, canonically polarized varieties, and singularities. We also consider the Semi-Stable Reduction Theorem and results concerning simultaneous stable reduction.

INTRODUCTION

The Deligne–Mumford [44] stable reduction theorem asserts that given a family of stable curves over a punctured disk, after a finite base change, the family can be completed to a family of stable curves over the disk. Moreover, the central fiber of the new family is determined up to isomorphism by the original family. This theorem plays a central role in the study of curves. One of the main consequences is the fundamental result that the moduli space of stable curves is compact. More qualitatively, the theorem provides control over degenerations of smooth curves: when studying one-parameter degenerations, one may restrict to the case where the limit is a stable curve.

In [62, Section 3.C], Harris–Morrison give a beautiful treatment of the stable reduction theorem from a computational perspective. They outline a proof of the theorem that provides the reader with a method of completing this process in particular examples, and importantly, of identifying the central fiber of the new family. The aim of this survey is to complement what is in Harris–Morrison with stable reduction problems in other settings.

Stable reduction problems arise in numerous situations. Roughly speaking, by a stable reduction problem we mean the problem of extending a family over the punctured disk to a family over the disk, by adding a “reasonable” fiber over 0. This may require a finite base change, and the fiber over 0 should be determined up to isomorphism by the original family. Specifying what is meant by a family, or by a “reasonable” fiber, is often an important part of the problem.

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Typically the motivation will be a moduli problem, where one is given a particular class of geometric or algebraic objects that determine a non-compact moduli space. The stable reduction problem can be viewed as providing a modular compactification of this space. In the language of stacks, stable reduction is equivalent to the valuative criterion of properness for the moduli stack.

It is not always immediately apparent how to extend the moduli problem to obtain a compact moduli space. In this situation one can begin with the more qualitative goal of obtaining some level of control over degenerations. One may, for instance, focus on restricting the singularities of the central fiber, or on improving other invariants such as monodromy. We will generally refer to these types of problems as semi-stable reduction problems.

Finally, it may be the case that there are different classes of objects that provide stable reduction theorems. For instance, in addition to the moduli space of stable curves \overline{M}_g , there are other modular compactifications of the moduli space of smooth curves. Recently, in light of the Hassett–Keel program, these spaces have received renewed interest. Stable reduction theorems in this setting can be viewed as describing resolutions of the birational maps among the moduli spaces.

In light of the breadth of the topic, to prevent this survey from becoming too lengthy, we have chosen to focus on a few cases that have some historic connection to the stable reduction theorem for curves, capture the flavor of the topic in general, and which point to some of the recent progress in the field. We also include a number of examples. In the end, the material chosen reflects the author’s exposure to the subject, and he apologizes to those people whose work was not included.

Outline. The first section is a worked example of a (simultaneous) stable reduction for a family of elliptic curves. The example can be viewed in a number of different ways, and provides explicit motivation for many of the topics that come later. In §2, we review the monodromy representation, which plays a central role in stable reduction for curves and abelian varieties. In §3, we focus on stable reduction for abelian varieties. Historically, this is one of the first instances where stable reduction was considered, and stable reduction for abelian varieties was used to give the original proof of the stable reduction theorem for curves in positive characteristic. This section culminates with a review of the recent work of Alexeev [8] providing a proper moduli space compactifying the space of principally polarized abelian varieties. The next section, §4, concerns stable reduction for curves. The focus is on the connection between the case of curves and the case of abelian varieties.

In §5, we consider the Semi-stable Reduction Theorem of Mumford et al. [71], which states that in characteristic zero, a one-parameter family of smooth schemes can be filled in (after a finite base change) to a family with central fiber a normal crossing divisor. The proof of stable reduction

in Harris–Morrison essentially proceeds via this theorem, and parts of the proof of the general case are quite accessible. Moreover, the Semi-stable Reduction Theorem has been used recently in many settings to arrive at stable reduction theorems for higher dimensional varieties. For instance in §5.3, we sketch a part of Kollár’s proof [75] of a stable reduction theorem for canonically polarized varieties that uses the Semi-stable Reduction Theorem.

Another situation where there has been recent progress is in the case of simultaneous stable reduction. This is the case where one allows the base of the family to have dimension greater than 1. In each of the sections mentioned above, we discuss the related simultaneous stable reduction problem. For instance, in §2, we discuss the work of Faltings–Chai [47] on semi-abelian varieties, and the connection to the Borel Extension Theorem [25]. We also provide a statement on simultaneous stable reduction for Alexeev’s space; this is perhaps the only result not appearing explicitly in other sources (although we expect it is well known to the experts). For curves, in §3, we discuss the recent simultaneous stable reduction theorems due to de Jong [42], de Jong–Oort [43], and Cautis [36]. Connections with the Hassett–Keel program are used to motivate a discussion of explicit simultaneous stable reduction for curves with ADE singularities, due to Laza and the author [35], and independently to Fedorchuk [50] for AD singularities. Finally, in §5.2, we consider the results of Abramovich–Karu [3] on simultaneous semi-stable reduction for higher dimensional varieties.

Semi-stable reduction theorems for singularities play an important role in understanding stable reduction for varieties. We include a section on this topic (§6). This includes a brief discussion of the well known result of Brieskorn [29] on simultaneous resolution of ADE surface singularities, as well as a discussion of some of the recent results of Hassett [63] on local stable reduction for curves. Finally, many moduli spaces are constructed as GIT quotients. We discuss in §7 a stable reduction statement in GIT that sheds light on the stable reduction problem for moduli spaces arising in this way.

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Notation and conventions.

1. A **family of schemes** $f : X \rightarrow B$ will be a flat, surjective, finite type morphism of schemes, of constant relative dimension. The scheme B will be called the **base** of the family and X the **total space** of the family. For a point $b \in B$, we denote by X_b the fiber of f over b .

2. We will typically use the following notation for spectrums of discrete valuation rings (DVRs). For a DVR R we will use the notation $K = K(R)$ for the fraction field, and $\kappa = \kappa(R)$ for the residue field. We will set $S = \operatorname{Spec} R$, with generic point $\eta = \operatorname{Spec} K$ and closed point $s = \operatorname{Spec} \kappa$. We will occasionally work in the analytic category, where S will denote the open unit disk in the complex plane, and S° the punctured unit disk.

3. If B is noetherian, and the family $f : X \rightarrow B$ is of constant relative codimension d , we define the **discriminant**, denoted Δ , to be the scheme theoretic image of the d -th fitting scheme of the coherent sheaf $\Omega_{X/B}$. The discriminant parameterizes the singular fibers of the family. Typically, we consider this in the case where either the d -th fitting scheme of $\Omega_{X/B}$ is finite over B , or f is proper; in these cases Δ is a closed subscheme of B .

4. Let X be a scheme over an algebraically closed field k , which is regular in codimension one, and let D be an effective Weil divisor on X . We say D is in **étale (resp. Zariski or simple) normal crossing** position if X is regular along the support of D and for each closed point $x \in \operatorname{Supp}(D)$ there exists an étale morphism (resp. an open inclusion) $f : U \rightarrow X$ such that for any $u \in U$ with $f(u) = x$, there is a local system of parameters x_1, \dots, x_n for $\mathcal{O}_{U,u}$ so that the pull-back of D via the composition $\operatorname{Spec} \mathcal{O}_{U,u} \rightarrow U_x \rightarrow X$, is defined by a product $x_1^{n_1} \cdots x_r^{n_r}$, for some $0 \leq r \leq n$ and some natural numbers n_1, \dots, n_r . We will say a divisor is **nc**, (resp. **snc**) if it is in étale normal crossing (resp. simple normal crossing) position.

5. A **modification** is a proper, birational morphism. An **alteration** is a generically finite, proper, surjective morphism.

6. A **germ** will be the spectrum of a complete local ring A and we will use the notation (X, x) with $X = \operatorname{Spec} A$ and x the maximal idea of A . A **(germ of a) singularity** will be a germ that is singular at x . We will typically focus on **hypersurface singularities**; by this we mean the case where $X = \operatorname{Spec} k[[x_1, \dots, x_n]]/(f)$, $f \in k[[x_1, \dots, x_n]]$ and k is a field. We will say a singularity is **isolated** if $\mathcal{O}_{X,x'}$ is a regular local ring for all $x' \in X$ with $x' \neq x$.

7. We use the notation $M_{g,n}$ for the moduli space of n -pointed, non-singular curves, and $\overline{M}_{g,n}$ for the Deligne–Mumford compactification. In general, for a moduli problem, we will use calligraphic symbols (\mathcal{M}) for the associated moduli stack, script symbols (\mathscr{M}) for the associated moduli functor, and Roman symbols (M) for the associated coarse moduli space.

1. AN EXAMPLE VIA ELLIPTIC CURVES

We start by considering a family of elliptic curves degenerating to a plane curve with a cusp. The presentation we give is a special case of a larger example described by Laza and the author in [35] (related to well known work of Brieskorn [28] and Tyurina [101, §3]; see also the recent work of

Fedorchuk [49, §5], [50]) and can be viewed as an extension of the discussion in Harris-Morrison [62, p.129-30].

1.1. The family of elliptic curves. Fix an algebraically closed field k with characteristic not equal to 2 or 3. Our starting point is the family of curves

$$x_2^2 + x_1^3 + t_2 x_1 + t_3 = 0$$

with parameters t_2 and t_3 . We denote the family by $X \rightarrow B$. One can easily check that the curve defined by the point (t_2, t_3) is non-singular if and only if $4t_2^3 - 27t_3^2 \neq 0$. The curve has a unique singularity, which is a node, if $4t_2^3 - 27t_3^2 = 0$ and $(t_2, t_3) \neq (0, 0)$, and the curve has a unique singularity, which is a cusp, if $(t_2, t_3) = (0, 0)$. Despite the family technically being none of the following, we will view it simultaneously as a family of projective curves of arithmetic genus one, a degenerate family of abelian varieties, and a deformation of a cusp.

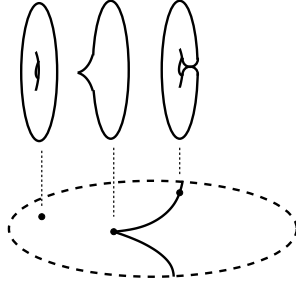


FIGURE 1. A degenerate family

Remark 1.1. To make this discussion precise we should take

$$X = \text{Proj}_{\mathbb{A}_k^2} \left(\frac{k[t_2, t_3][X_0, X_1, X_2]}{(X_0 X_2^2 + X_1^3 + t_2 X_0^2 X_1 + X_0^3 t_3)} \right) \subseteq \mathbb{P}_k^2 \times \mathbb{A}_k^2,$$

$B = \mathbb{A}_k^2 = \text{Spec } k[t_2, t_3]$, $\pi : X \rightarrow \mathbb{A}_k^2$ the morphism induced by the second projection $\mathbb{P}_k^2 \times \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$, and $\sigma_\infty : \mathbb{A}_k^2 \rightarrow X$ the section at infinity given by the ring homomorphism

$$\frac{k[t_2, t_3][X_0, X_1, X_2]}{(X_0 X_2^2 + X_1^3 + t_2 X_0^2 X_1 + X_0^3 t_3)} \rightarrow k[t_2, t_3][X_0]$$

defined by the ideal (X_1, X_2) . We then obtain a diagram

$$(1.1) \quad \begin{array}{ccc} X & \hookrightarrow & \mathbb{P}_k^2 \times \mathbb{A}_k^2 \\ & \searrow \pi & \downarrow \pi_2 \\ & & \mathbb{A}_k^2 \\ & \nearrow \sigma_\infty & \end{array}$$

The morphism $\pi : X \rightarrow \mathbb{A}_k^2$ is a flat family of projective curves of arithmetic genus one. The section σ_∞ defines a group scheme structure and

polarization on the generic fiber. This makes the generic fiber a principally polarized abelian scheme of dimension one. Restricting to germs, we obtain a deformation of a cusp.

Let us make a few more informal observations. Set

$$G = X - \{(0, 0, t_2, t_3) : 4t_2^3 - 27t_3^2 = 0\}$$

(where here we are taking X to be the projective family). Then $\pi : G \rightarrow \mathbb{A}_k^2$ is a family of commutative groups. The group parameterized by (t_1, t_2) is a copy of \mathbb{G}_m , if $4t_2^3 - 27t_3^2 = 0$ and $(t_2, t_3) \neq (0, 0)$. The group is a copy of \mathbb{G}_a , if $(t_2, t_3) = (0, 0)$. These are the groups of line bundles of degree zero on the corresponding fibers. In fact G/\mathbb{A}_k^2 is the relative (connected component of the) Picard scheme $\mathbf{Pic}_{X/\mathbb{A}_k^2}^0$, and X/\mathbb{A}_k^2 is the compactified (connected component of the) Picard scheme $\overline{\mathbf{Pic}}_{X/\mathbb{A}_k^2}^0$ (see Altman–Kleiman [14]).

For the purpose of this discussion, we view it as pathological that the central fiber of the family $\pi : X \rightarrow \mathbb{A}_k^2$ has a cusp (and the central fiber of the family $\mathbf{Pic}_{X/\mathbb{A}_k^2}^0 \rightarrow \mathbb{A}_k^2$ is an additive group). Our goal will be to modify the family so that we may replace the central fiber with a nodal curve (or a copy of \mathbb{G}_m in the case of the family of groups, or a collection of smooth components meeting transversally in the case of a singularity).

The problem can also be stated in stack-theoretic language. Let $\overline{\mathcal{M}}_{1,1}$ be the moduli stack of Deligne–Mumford stable, one-pointed curves of arithmetic genus one, and let $\overline{M}_{1,1}$ be the coarse moduli space. The family X/\mathbb{A}_k^2 defines a rational map $\mathbb{A}_k^2 \dashrightarrow \overline{\mathcal{M}}_{1,1}$ and we would like to give a resolution of this map.

1.2. Explicit stable reduction. We now construct an explicit simultaneous stable reduction of the family. We will do this in several steps, and then discuss a monodromy computation that sheds light on the problem.

1.2.1. Step 1: pulling back by a Weyl (group) cover. Consider the map

$$\begin{aligned} \{a_1 + a_2 + a_3 = 0\} &\rightarrow \mathbb{A}_k^2 \\ (a_1, a_2, a_3) &\mapsto (a_1a_2 + a_1a_3 + a_2a_3, -a_1a_2a_3). \end{aligned}$$

The families obtained are given by the diagram below.

$$(1.2) \quad \begin{array}{ccc} \{a_1 + a_2 + a_3 = x_2^2 + \prod_{i=1}^3 (x_1 - a_i) = 0\} & \longrightarrow & \{x_2^2 + x_1^3 + t_2x_1 + t_3 = 0\} \\ \downarrow & & \downarrow \\ \{a_1 + a_2 + a_3 = 0\} & \longrightarrow & \mathbb{A}_k^2. \end{array}$$

There is still a unique fiber that is cuspidal, but the discriminant has been replaced by a hyperplane arrangement of type A_2 , given by the equation

$$\{(a_2 - a_3)^2(a_1 - a_3)^2(a_1 - a_2)^2 = 0\}.$$

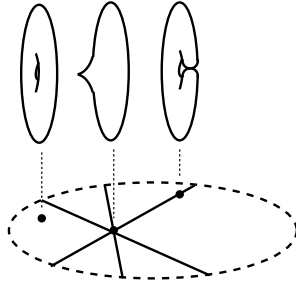


FIGURE 2. The Weyl cover

Set $B' \rightarrow B$ to be the finite (Weyl) cover defined above, and set $X' \rightarrow B'$ to be the family obtained by pull-back. The Weyl group in this case is the group of type A_2 ; i.e. the permutation group Σ_3 .

1.2.2. *Step 2: a wonderful blow-up.* It is a general principle that putting the discriminant locus into normal crossing position is beneficial (not only is a normal crossing divisor easier to understand, there is also the Borel Extension Theorem [25] for abelian varieties and the work of de Jong–Oort [43] and Cautis [36] for stable curves, all of which will be discussed in more detail in §3 and §4).

We put the discriminant in this example into nc position by blowing up the point that is the intersection of its three components. Explicitly, on one coordinate patch, we consider the map

$$\begin{aligned} \{1 + b_2 + b_3 = 0\} &\rightarrow \{a_1 + a_2 + a_3 = 0\} \\ (b_1, b_2, b_3) &\mapsto (b_1, b_1 b_2, b_1 b_3). \end{aligned}$$

Pulling the family back by this map gives the new family:

$$\begin{aligned} (1.3) \quad \{1 + b_2 + b_3 = x_2^2 + (x_1 - b_1) \prod_{i=1}^2 (x_1 - b_1 b_i) = 0\} &\longrightarrow \dots \\ \downarrow & \\ \{1 + b_2 + b_3 = 0\} &\longrightarrow \dots \end{aligned}$$

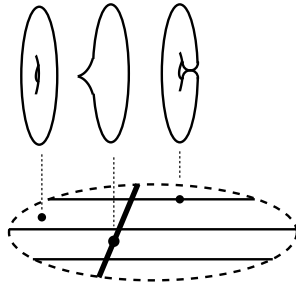


FIGURE 3. The wonderful blow-up

Denote the space obtained by this blow-up as $\tilde{B} \rightarrow B'$. We call this the wonderful blow-up. Let $\tilde{X} \rightarrow \tilde{B}$ be the family obtained by pull-back. This restricts to a family of cuspidal curves over the exceptional curve $\{b_1 = 0\}$. Note that the generic point of each irreducible component of the discriminant now parameterizes curves with a unique singularity of type A_1 or A_2 .

Remark 1.2. We will see below in §1.3 that over \tilde{B} we can not replace the cuspidal curves with stable curves. In other words, there does not exist a morphism extending the rational map $\tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$. However, we can extend the map to the moduli scheme (see §1.2.6); i.e. there is a morphism extending the rational map $\tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$.

1.2.3. *Step 3: a double cover.* In order to obtain a family of stable curves, we will need to take a double cover of the base, branched along the exceptional locus. The double cover is not possible globally (the exceptional divisor does not admit a square root), so we proceed locally. Consider the map

$$\begin{aligned} \{1 + c_2 + c_3 = 0\} &\rightarrow \{1 + b_2 + b_3 = 0\} \\ (c_1, c_2, c_3) &\mapsto (c_1^2, c_2, c_3). \end{aligned}$$

Pulling the family back by this map gives the new family:

$$(1.4) \quad \begin{array}{ccc} \{1 + c_2 + c_3 = x_2^2 + (x_1 - c_1^2) \prod_{i=1}^2 (x_1 - c_1^2 c_i) = 0\} & \longrightarrow & \dots \\ \downarrow & & \\ \{1 + c_2 + c_3 = 0\} & \longrightarrow & \dots \end{array}$$

Let us denote this finite cover by $\tilde{B}' \rightarrow \tilde{B}$ and let $\tilde{X}' \rightarrow \tilde{B}'$ be the family obtained by pull-back.

1.2.4. *Step 4: blowing up the cusp locus in the total space.* There is a family of cuspidal curves lying over the locus $\{c_1 = 0\}$. In the total space \tilde{X}' , the locus of cusps in the fibers is given as $\{c_1 = x_1 = x_2 = 0\}$. Our goal will be to perform a blow-up supported on this locus that will provide a family of semi-stable curves.

To do this, blow-up \tilde{X}' along the ideal

$$I = ((c_1^2, x_1)^3, (c_1^3, c_1 x_1) \cdot (x_2), x_2^2).$$

Let us denote the resulting family as $\text{Bl}_I \tilde{X}' \rightarrow \tilde{B}'$. The blow-up replaces the cuspidal curves with nodal curves consisting of two irreducible components: the desingularization of the cuspidal curve, which is a copy of \mathbb{P}^1 sitting inside of the blow-up $\text{Bl}_{(x_1^3, x_2^2)} \mathbb{A}_k^2$, and a stable elliptic curve sitting inside of the weighted projective space $\mathbb{P}(1, 2, 3)$. We mention here that Hassett [63, §6.2] has determined the tails arising from a much more general class of singularities; we will discuss these results later in §6.

In short, we have locally (on the base) constructed an explicit semi-stable reduction of the cuspidal family, which is stable except in the fibers over the locus $\{c_1 = 0\}$, where it is nodal, but not stable.

1.2.5. *Step 5: the relative dualizing sheaf.* Finally, one can take the relative canonical model (obtained via the relative dualizing sheaf) for the family of nodal curves. Concretely, this will contract the extraneous \mathbb{P}^1 s in the fibers, giving a family of stable curves. Let us denote this family by $\widehat{X} \rightarrow \widetilde{B}'$.

1.2.6. *Summary.* We now have a family $\widehat{X} \rightarrow \widetilde{B}'$ of stable curves extending the pull-back of the original family, where \widetilde{B}' is an alteration of B . By

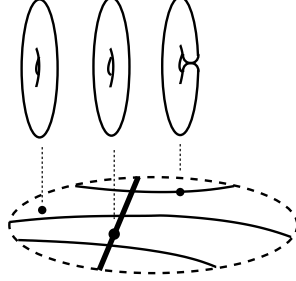


FIGURE 4. The simultaneous stable reduction

definition, we obtain a morphism

$$\widetilde{B}' \rightarrow \overline{\mathcal{M}}_{1,1} \rightarrow \overline{M}_{1,1}.$$

The map $\widetilde{B}' \rightarrow \widetilde{B}$ is finite, so in fact there is a map $\widetilde{B} \rightarrow \overline{M}_{1,1}$ (e.g. [36, Lem. 2.4]). The locus $\{c_1 = 0\}$ can be identified with the λ -line, with corresponding family given by $y^2 = x(x-1)(x-\lambda)$. The points $\lambda = 0, 1, \infty$ correspond to the intersections of the strict transforms of the hyperplane arrangement (the discriminant after the Weyl cover). The restricted map $\{c_1 = 0\} \rightarrow \overline{M}_{1,1}$ can be identified with the map from the λ -line to the j -line.

1.3. Obstructions. We have seen that there exists an extension of the rational map $\widetilde{B} \dashrightarrow \overline{M}_{1,1}$ to the moduli scheme $\widetilde{B} \rightarrow \overline{M}_{1,1}$. We now show the rational map $\widetilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ to the moduli stack does not extend; i.e. there is no family of stable curves over \widetilde{B} extending the pull-back of the original family.

We do this in the following way. Let S be the spectrum of a DVR with closed point s and generic point η . We will find a morphism $S \rightarrow \widetilde{B}$ sending s to a closed point of the exceptional divisor (i.e. $\{c_1 = 0\}$, parameterizing the cuspidal locus) and sending η to the generic point of \widetilde{B}' (i.e. the smooth locus). Then we will show that the induced family of curves $X_S \rightarrow S$ does not extend to a family of stable curves; i.e. the composition $S \rightarrow \widetilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ does not extend to a morphism.

Remark 1.3. It was pointed out by Fedorchuk [50, Prop. 7.4] that this in fact shows the stronger statement that the rational map $B \dashrightarrow \overline{\mathcal{M}}_{1,1}$ can

not be resolved over any alteration $A \rightarrow B$ that is isomorphic to B' over the locus $B - \Delta$.

We will show in two ways that the general $S \rightarrow \tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ as above does not extend to a morphism. The first is via a monodromy computation. The second method is via a computation following an argument of Fedorchuk [50].

1.3.1. The monodromy obstruction. Consider the family $X' \rightarrow B'$ obtained via the Weyl cover, and the restriction $(X')|_L \rightarrow L$ of this family to a generic line L through the origin in B' . To show that there is no extension $\tilde{B} \rightarrow \overline{\mathcal{M}}_{1,1}$ to the moduli stack, it suffices to show that the restriction $(X')|_L \rightarrow L$ does not extend to a stable family of curves. To show this, observe that in the notation of §1.2.2, the restriction $(X')|_L \rightarrow L$ is a surface Z_{b_2} with equation (locally near the A_2 singularity):

$$(1.5) \quad x_2^2 + x_1^3 - (b_2^2 + b_2 + 1)b_1^2x_1 - b_2(1 + b_2)b_1^3 = 0,$$

where b_1 is a parameter for L and b_2 is a fixed slope.

The surface Z_{b_2} has a D_4 singularity at the origin. This is also a cusp singularity for X_0 , the central fiber of Z_{b_2} viewed as a family of curves. Recall that the standard resolution of a D_4 surface singularity $x^2 = f_3(y, z)$ is given by 4 blow-ups: First blow-up the D_4 singularity. This gives an exceptional divisor E_0 . The D_4 singularity “splits” into three A_1 singularities corresponding to the three roots of f_3 . Then blow-up each A_1 singularity. This introduces exceptional divisors E_1, E_2, E_3 , giving the desired resolution. We associate to this a \tilde{D}_4 graph (consisting of E_0 the central vertex, to which one attaches edges connecting the 4 vertices corresponding to the curves X_0, E_1, E_2 and E_3).

The monodromy obstruction can be identified via the theory of elliptic fibrations. From the \tilde{D}_4 graph, we conclude that this is a type I_0^* degeneration in Kodaira’s classification (see [21, §V.7, p.201]). It follows that the monodromy is $-\text{Id}$ (see [21, p.210]). We also direct the reader to the discussion of the elliptic involution in [62, Ch. 2A], and to the monodromy computation made in §2.1.2. In conclusion, the monodromy not being unipotent, the family does not extend to a family of stable curves (this fact is reviewed in §3 and §4).

1.3.2. An obstruction via a direct computation. Again, our goal is to show that the general map $S \rightarrow \tilde{B} \dashrightarrow \overline{\mathcal{M}}_{1,1}$ with closed point sent to the cuspidal locus, and generic point sent to the smooth locus, does not extend to a morphism.

We follow an observation of Fedorchuk [50, Prop. 7.4]. Let S' be the spectrum of a DVR, which is a branched double cover of S admitting a morphism to \tilde{B}' . Pulling back the family $\hat{X} \rightarrow \tilde{B}'$ we obtain a family of stable curves $\hat{X}_{S'} \rightarrow S'$. Fedorchuk’s observation is that it suffices to show that the total space $\hat{X}_{S'}$ is smooth. Indeed, if there were a family of stable

curves $\widehat{X}_S \rightarrow S$ extending the pull-back of the original family, then it would follow that $\widehat{X}_{S'}$ was equal to $S' \times_S \widehat{X}_S$. But then the total space $\widehat{X}_{S'}$ would have a singular point, giving a contradiction (the point of \widehat{X}_S at the node of the central fiber, locally given by $xy - t^n$ with $n \geq 1$, would be replaced with a singular point $xy - t^{2n}$ of $\widehat{X}_{S'}$).

Fedorchuk's approach is to construct a particular one-parameter family of genus two curves degenerating to a cusp (his argument implies the result for families of curves of arbitrary genus [50, Prop. 7.4]). Alternatively, with the work we have done here in coordinates, one can show that the blow-up in the fourth step gives a smooth total space when restricted to the general S' . Since the total space is a smooth surface, and all of the curves blown down in the fifth step are (-1) -curves, this does not introduce singularities in the total space.

2. MONODROMY

The monodromy representation is a topological invariant associated to a family over a punctured disk. In this section, we briefly review the definition of monodromy, and then compute a few examples. We then state the monodromy theorem. While there is an algebraic monodromy representation for families over arbitrary DVRs, for simplicity, we restrict to the case of monodromy in the analytic setting.

2.1. Preliminaries on monodromy. Let $X^\circ \rightarrow S^\circ$ be a smooth family of complex, projective varieties over the punctured disk. It is well known that for each $t_1, t_2 \in S^\circ$, the fiber X_{t_1} is diffeomorphic to the fiber X_{t_2} (see e.g. [72, Thm. 2.3, p.61]). In particular, the fibers are all homeomorphic, and the cohomology groups $H^\bullet(X_t, \mathbb{C})$ are isomorphic for all $t \in S^\circ$. Fix a base point $* \in S^\circ$ and consider a path $\gamma : [0, 1] \rightarrow S^\circ$ that generates $\pi_1(S^\circ, *)$. The family of groups $H^\bullet(X_{\gamma(\tau)}, \mathbb{C})$, $\tau \in [0, 1]$, determines an automorphism of $H^\bullet(X_*, \mathbb{C})$. The induced homomorphism

$$\pi_1(S^\circ, *) \rightarrow \text{Aut } H^\bullet(X_*, \mathbb{C})$$

is called the (analytic) monodromy representation of the family.

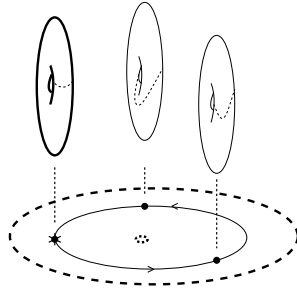


FIGURE 5. Monodromy

For a family that extends to a smooth family over S , the homomorphism is trivial. Recall that an endomorphism T of a finite-dimensional vector space V is said to be **unipotent** (resp. **quasi-unipotent**) if there exist $M \geq 1$ (resp. $M, N \geq 1$) such that $(T - \text{Id}_V)^M = 0$ (resp. $(T^N - \text{Id}_V)^M = 0$).

2.1.1. *A family of stable curves.* Consider the family

$$x_2^2 - (x_1^2 - t)(x_1 - 1);$$

i.e. a family of smooth elliptic curves degenerating to a nodal cubic. Set $* = 1/2$ and let $\gamma : [0, 1] \rightarrow S^\circ$ be a parameterization of the circle of radius $1/2$. The family of varieties lying over γ is a family of elliptic curves determined by the branch locus $\{-\sqrt{t}, \sqrt{t}, 1, \infty\}$. There is a basis of $H^1(X_*, \mathbb{C})$ for which the monodromy representation is given by

$$M_{A_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

See Carlson–Müller–Stach–Peters [31, p.18] for a detailed exposition of this. Note that this matrix is unipotent. This transformation is in fact a special case of a more general phenomenon, described by the Picard–Lefschetz theorem (see e.g. [15, Ch.2, §1.5]).

2.1.2. *A family of cuspidal curves.* Consider the family

$$x_2^2 - x_1^3 - t;$$

i.e. a family of smooth elliptic curves degenerating to a cuspidal cubic. Again, set $* = 1/2$ and let $\gamma : [0, 1] \rightarrow S^\circ$ be a parameterization of the circle of radius $1/2$. There is a basis of $H^1(X_*, \mathbb{C})$ for which the monodromy representation is given by

$$M_{A_2} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Figure 6 shows the transformation of cycles on the copy of \mathbb{P}^1 lying below

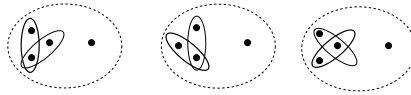


FIGURE 6. Monodromy for a cuspidal family

the elliptic curve, with respect to the branch locus, for $t = \gamma(0) = 1/2$, $t = \gamma(1/2) = -1/2$ and $t = \gamma(1) = 1/2$, respectively. Considering lifts of these cycles, one arrives at the matrix above.

In this example, the monodromy representation is quasi-unipotent, but not unipotent. Note additionally that $M_{A_2}^3 = -\text{Id}$. This implies that after pulling the family back by a triple cover, the monodromy will be given by $-\text{Id}$. This new family gives a special case of the monodromy obstruction computation made in §1.3.1; in the notation of that section, this is the family given by taking $t_2 = 0$.

Note further that since $M_{A_2}^6 = \text{Id}$, if we pull the family back by a six-fold cover, the monodromy becomes trivial. This can be seen directly in the following way. The family obtained after a six-fold cover is

$$x_2^2 - x_1^3 - t^6.$$

Changing coordinates by $x_1 \mapsto x_1 t^2$ and $x_2 \mapsto x_2 t^3$ gives the family

$$x_2^2 - x_1^3 - 1.$$

In other words, after the degree six base change, the family can be extended to a *trivial* family over S .

2.2. The monodromy theorem. The monodromy theorem is a general statement about the monodromy representation of a family of projective manifolds over the punctured disk. This will play an important role in regards to an extension theorem of Grothendieck's for abelian varieties, which we discuss in §3.

Theorem 2.1 (Monodromy Theorem). *Let $\pi^\circ : X^\circ \rightarrow S^\circ$ be a family of smooth, complex, projective manifolds of dimension n over the punctured disk. For each integer $0 \leq k \leq 2n$, the monodromy representation*

$$\pi_1(S^\circ, *) \rightarrow \text{Aut } H^k(X_*, \mathbb{C})$$

is quasi-unipotent.

The reader is directed to Griffiths [57, Rem. 3.2, p.236] for references, including a discussion of the history of the theorem and a description of the many different methods of proof (see also Grothendieck [97, Thm. 1.2, p.6] for the algebraic statement).

Remark 2.2. The monodromy theorem implies that for a family of smooth, complex, projective manifolds over the punctured disk, after a finite base change the monodromy can be made unipotent. Indeed, if the generator of the monodromy representation is given by the automorphism T , then $(T^N - \text{Id})^M = 0$ for some N, M . Thus after the base change given by $t \mapsto t^N$, the monodromy will be unipotent. We note that many of the proofs of the monodromy theorem provide bounds on N and M .

Remark 2.3. If $\pi : X \rightarrow S$ is a generically smooth family of complex projective varieties, such that $X_0 := \pi^{-1}(0)$ is an snc divisor in X , then the monodromy representation is unipotent (see the references in [57]). One can deduce the Monodromy Theorem from this using the Semi-Stable Reduction Theorem [71] (see §5.1).

Remark 2.4. As is evident in the previous remark, if $\pi : X \rightarrow S$ is a generically smooth family of complex projective varieties, the topology of X_0 is related to the monodromy of the family. The Clemens–Schmid exact sequence makes this precise (e.g. Morrison [84, p.109]). There is also a notion of vanishing cohomology for isolated singularities on X_0 . There is a monodromy operator on the vanishing cohomology, which is related to the

monodromy of the family by an exact sequence. We direct the reader to [97, pp.V, 79] and [100, (1.4)] for more details.

3. ABELIAN VARIETIES

One of the first places where questions about stable reduction were considered was for abelian varieties. The family described in §1, viewed as a family of abelian varieties, gives a concrete example. Historically, one of the motivations for the development of the theory was to study abelian schemes over \mathbb{Q} by reducing modulo a prime p . Viewing the abelian scheme over \mathbb{Q} as a family over the generic point of $\mathrm{Spec} \mathbb{Z}$, problems concerning reduction modulo a prime can be translated into problems about extending abelian schemes over \mathbb{Q} to schemes over $\mathrm{Spec} \mathbb{Z}$.

While a well known theorem of Fontaine [51, Cor., p.517] states there are no abelian schemes over $\mathrm{Spec} \mathbb{Z}$, so there can not be an extension to an abelian scheme over every prime, the stable reduction theorem addresses the question of extending over a particular prime after a finite base change. With this as motivation, we start this section with an example with an arithmetic flavor.

Another motivation for considering stable reduction for abelian varieties is that it was used in the original proof of the stable reduction theorem for curves in positive characteristic. We discuss this further in the next section. In this section we also consider stable reduction in the context of Alexeev's moduli space of stable semiabelic pairs [8]. Finally, we consider the question of simultaneous stable reduction and the results of Faltings–Chai [47].

3.1. An example of stable reduction for a family of abelian varieties. The following arithmetic example is closely related to the previous geometric examples, and emphasizes the connection between the two settings. We will use the terminology of group schemes, which we review in the next subsection.

Let $X \rightarrow \mathrm{Spec} \mathbb{Z}$ be the projective scheme defined by

$$y^2 - x^3 - 25\alpha x - 125\beta = 0,$$

with α and β integers such that $4\alpha^3 + 27\beta^2$ is not divisible by 5. Let $X_{\mathbb{Q}}$ be the scheme obtained by base change to $\mathrm{Spec} \mathbb{Q}$. There is the usual group law on $X_{\mathbb{Q}}$ induced by the point at infinity $(0 : 1 : 0)$. We are interested in understanding how this fails to extend to a group law on X over $\mathrm{Spec} \mathbb{Z}$, and how one might attempt to rectify this at a particular prime by taking a finite cover.

Concerning the group law, one can check that $X \rightarrow \mathrm{Spec} \mathbb{Z}$ fails to be smooth over the primes 2, 3 and 5, so that $X_{\mathbb{Q}}$ does not extend to a group scheme over those points. In this example, we focus on the issue of extension over (5). Let

$$X_{(5)} \rightarrow \mathrm{Spec} \mathbb{Z}_{(5)}$$

be the scheme obtained from X by base change. The fiber over the generic point (0) is $X_{\mathbb{Q}}$ and we are interested in extending $X_{\mathbb{Q}}$ to an abelian scheme over $\text{Spec } \mathbb{Z}_{(5)}$.

Let $X_{\mathbb{F}_5}$ be the fiber of $X_{(5)}$ over the closed point. Then $X_{\mathbb{F}_5}$ is given by the equation

$$y^2 - x^3 = 0,$$

which is singular at $(0 : 0 : 1)$. We would like to describe a finite base change and a modification of the family that is smooth.

Consider the finite, degree 2 morphism

$$B' := \text{Spec } \mathbb{Z}_{(5)}[\zeta]/(\zeta^2 - 5) \rightarrow \text{Spec } \mathbb{Z}_{(5)}$$

induced by the extension $\mathbb{Q}(\sqrt[2]{5})/\mathbb{Q}$. Pulling back $X_{(5)}$ we obtain a family $X' \rightarrow B'$ defined by the equation

$$y^2 - x^3 - \alpha\zeta^4x - \beta\zeta^6 = 0.$$

Making the change of coordinates $x \mapsto \zeta^2x$, $y \mapsto \zeta^3y$, we arrive at a family $\tilde{X} \rightarrow B'$ defined by

$$y^2 - x^3 - \alpha x - \beta = 0.$$

This is smooth over the closed point $(\zeta) \in B'$, and in fact there is a group law over B' . Thus, after a finite, degree two, base change, we have modified our family to give an abelian scheme over the base.

Remark 3.1. It is interesting to note the connection with the geometric case. The example above was constructed to be the analogue of the linear family over the Weyl cover (§1.3.1):

$$y^2 + x^3 - (b_2^2 + b_2 + 1)b_1^2x - b_2(1 + b_2)b_1^3 = 0,$$

where, roughly speaking, we replaced x_2 with y , x_1 with $-x$, set $b_1 = 5$ and took b_2 general. We had seen that a degree two base change for the analytic family would allow for stable reduction, and this is exactly what we have found here in the arithmetic setting.

Remark 3.2. For completeness, we mention that the discriminant Δ and j -invariant of X are

$$\Delta = -16(4\alpha^3 + 27\beta^2)(5^6) \neq 0 \quad \text{and} \quad j = -1728(4\alpha)^3 5^6 / \Delta.$$

Note that j has non-negative valuation at 5. It is well known that from this data one can deduce that the family does not have abelian reduction at 5, but does have potentially abelian reduction there (see e.g. [99, VII Prop. 5.1, 5.5]).

3.1.1. Monodromy. Recall that the analytic monodromy representation of the analogous family was given by the negative of the identity (§1.3.1). Although we have not introduced the algebraic monodromy operator, we make the following observation. Since there is abelian reduction after a degree two base change, one can immediately conclude that the algebraic monodromy operator is a square root of the identity. One can then show the action on non-trivial, torsion points of sufficiently high order (relatively prime to 2 and 5) is non-trivial. Thus the monodromy operator is the negative of the identity, similar to the analogous analytic case.

3.2. Group scheme terminology. We now review some of the basic terminology of group schemes, directing the reader to [27, §4.1] and [88, Ch.6] for more details. For a scheme B , a **B -group scheme** is a group object in the category of B -schemes (Sch/B). A standard example, which we will use frequently, is $\mathbb{G}_m = \text{Spec } \mathbb{Z}[t, t^{-1}]$, with group law induced by the map

$$\mathbb{Z}[t, t^{-1}] \rightarrow \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \quad \text{given by} \quad t \mapsto t \otimes t.$$

For an arbitrary scheme B , we define $\mathbb{G}_{m,B}$ by base extension, and the induced group law makes $\mathbb{G}_{m,B}$ a group object in the category of B -schemes. An **(affine) split B -torus** T is a B -group scheme that is isomorphic as a B -group scheme to a finite fibered product $\mathbb{G}_{m,B} \times_B \dots \times_B \mathbb{G}_{m,B}$. An **(affine) B -torus** T is a B -group scheme that is étale locally on B a split torus. We define **B -subgroup schemes** in the obvious way (see e.g. [27, p.98]).

An **abelian scheme over B** is a B -group scheme that is smooth and proper over B with connected fibers. It follows from the Rigidity Lemma that the group law of an abelian scheme is commutative (see e.g. [88, Pro. 6.1, Cor. 6.4, p.115-6]). It is a well known result that an abelian scheme over a field is projective ([103]).

Remark 3.3. In order to use the term *variety* consistently (within this paper), we reserve the term **abelian variety** for an abelian group scheme over an algebraically closed field. (This is not standard, in that one usually does not require the field to be algebraically closed.)

The best understood abelian schemes are Jacobians of curves. Recall that associated to a smooth curve X over an algebraically closed field, there is an abelian variety JX , called the Jacobian of X , parameterizing degree zero line bundles on X . We note in addition that associated to a family of smooth curves $X \rightarrow B$, there is an associated abelian scheme JX_B over B , called the (relative) Jacobian of X_B , with geometric fibers that are the Jacobians of the associated curves.

A **semi-abelian scheme** G_B over B is a smooth, separated, commutative B -group scheme such that each fiber $G_{B,b}$ over $b \in B$ is an extension of an abelian scheme A_b by an affine torus T_b :

$$0 \rightarrow T_b \rightarrow G_{B,b} \rightarrow A_b \rightarrow 0.$$

We direct the reader to [47, Cor. 2.11] for a statement on the global structure of a semi-abelian scheme. Extensions of an abelian variety A/k by a torus T/k are classified (up to isomorphism as extensions) by $\text{Hom}(X(T), \hat{A})$, where $X(T) = \text{Hom}(T, \mathbb{G}_{m,k})$ is the character group and $\hat{A} = \text{Pic}^0(A)$ is the group of line bundles on A algebraically equivalent to zero (see e.g. [96, Thm. 6, p.184]).

We now come to the topic of reduction. Let R be a DVR, let K be its field of fractions, and let $S = \text{Spec } R$. Let A_K be an abelian scheme over $\text{Spec } K$. We say that A_K has **abelian (or good) reduction** (resp. **semi-abelian reduction**) if A_K extends to a smooth, separated S -group scheme G_S of finite type over S such that the fiber over the closed point $s \in S$ is an abelian (resp. semi-abelian) scheme over s . We will say that A_K has **potentially abelian (or good) reduction** (resp. **potentially semi-abelian reduction**) if there is a DVR R' with field of fractions K' and a finite morphism $\text{Spec } R' \rightarrow \text{Spec } R$ so that the abelian scheme $A_{K'}$ obtained by base change has abelian (resp. semi-abelian) reduction.

3.3. Néron models. Néron models provide a natural context for discussing the stable reduction theorem for abelian varieties. While the theory can be developed in more generality over Dedekind domains, we focus on the case of DVRs for simplicity.

As above, let $S = \text{Spec } R$ be the spectrum of a DVR with fraction field K . Let X_K be a smooth, separated K -scheme of finite type. A **Néron model** of X_K is an extension X_S of X_K over S that is a smooth, separated scheme of finite type, satisfying the following universal property: for any smooth S -scheme Y_S and any K -morphism $f_K : Y_K \rightarrow X_K$ there is a unique S -morphism $f_S : Y_S \rightarrow X_S$ extending f_K .

$$\begin{array}{ccccc}
 & & X_K & \xrightarrow{\quad} & X_S \\
 & \nearrow f_K & \downarrow & & \nearrow f_S \\
 Y_K & \xrightarrow{\quad} & Y_S & & \\
 & \searrow & \downarrow & & \searrow \\
 & & \text{Spec } K & \xrightarrow{\quad} & S
 \end{array}$$

$\exists!$

If a Néron model exists, it is unique up to unique isomorphism. While Néron models do exist in a more general setting, we will focus here on the case of abelian schemes. The main theorem in this situation is:

Theorem 3.4 (Néron [91]). *Let A_K be an abelian scheme over the field of fractions K of a DVR R . Then A_K admits a Néron model X_S over $S = \text{Spec } R$.*

We direct the reader also to [27, Cor. 2, p.16, Pro. 6, p.14] and Artin [17].

Remark 3.5. From the universal property of the Néron model, it follows that the K -group scheme structure on A_K extends uniquely to a commutative S -group scheme structure on X_S . For group schemes, the condition that the Néron model be of finite type and separated is superfluous (e.g. [27, p.12, Rem. 7, p.14]). Finally, it is a result of Raynaud that the Néron model of an abelian scheme is quasi-projective [94, Thm. VIII.2, p.120].

Remark 3.6. The special fiber of a Néron model of an abelian scheme need not be connected. One such example is given by a smooth plane cubic degenerating to the union of a quadric and line. The special fiber $X_{S,s}$ of the Néron model can be computed using a result of Raynaud discussed in the remark below. One can show $X_{S,s}$ fits into an exact sequence

$$0 \rightarrow \mathbb{G}_m \rightarrow X_{S,s} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$$

(see e.g. Kass [69, §4.3]). There is always, however, an open S -subgroup scheme of a Néron model of an abelian scheme A_K that extends A_K and has connected central fiber. We will denote this by X_S° .

Remark 3.7. As with abelian varieties, the best understood Néron models are those associated to curves. One of the main tools is a theorem of Raynaud's, relating the Néron model of a Jacobian to the Picard functor. The following is a weaker version of the theorem, given in Deligne–Mumford [44, Thm. 2.5], which is used in the proof of the stable reduction theorem for curves. *Assume the residue field of R is algebraically closed. In the notation above, let C_S be a generically smooth family of nodal curves over S , with non-singular total space, and let A_K be the Jacobian of the generic fiber. Then the open S -subgroup scheme X_S° of the Néron model X_S of A_K , described above, represents the relative (connected component of the) Picard functor.*

Example 3.8. Assume the residue field of R is algebraically closed. Consider a family $C \rightarrow S$ of smooth curves degenerating to an irreducible, stable curve C_s with a single node. Let C_s^ν be the normalization of C_s , and assume that C_s is obtained from C_s^ν by attaching points $p, q \in C_s^\nu$. The family of curves C_K over K determines a principally polarized abelian scheme $X_K = JC_K$, the Jacobian of the curve. The special fiber of the open S -subgroup scheme X_S° of the Néron model of X_K is an extension

$$0 \rightarrow \mathbb{G}_m \rightarrow X_{S,s}^\circ \rightarrow JC_s^\nu \rightarrow 0$$

determined by the data of the line bundle $\mathcal{O}_{C_s^\nu}(p - q)$.

3.3.1. The group structure of the central fiber of the Néron model. In the notation above, we have seen that the Néron model of an abelian scheme A_K is a commutative group scheme over S . To get a handle on how Néron models are connected to the question of semi-abelian reduction, we will investigate the group structure on the central fiber of the Néron model using a few basic facts from the theory of algebraic groups (see also Serre [96]).

To begin, we recall Chevalley's theorem [37, 95] (see esp. [27] and [39, Thm. 1.1]): *Let K be a field and let G be a smooth, connected algebraic K -group. Then there exists a smallest (not necessarily smooth) connected linear subgroup L of G such that the quotient G/L is an abelian scheme over K . Moreover, if K is perfect, L is smooth and its formation is compatible with change of base field.*

In other words, if X_S is the Néron model of A_K , then the connected component of the identity in the central fiber $X_{S,s}^\circ$ fits into an exact sequence over s

$$0 \rightarrow L \rightarrow X_{S,s}^\circ \rightarrow A \rightarrow 0,$$

where L is a connected, commutative, linear s -group scheme and A is an abelian s -group scheme.

We now turn our attention to the structure of linear algebraic groups. Let us pull-back to the algebraic closure $\bar{k} = \overline{\kappa(s)}$, and denote the resulting group schemes by \bar{L} , $\bar{X}_{S,s}^\circ$ and \bar{A} respectively. \bar{L} being commutative, it is solvable (see e.g. [26, Def., p.59]). There is the following standard theorem (e.g. [26, Thm. 10.6, p.137]): *For a connected, solvable, linear algebraic group \bar{L} over an algebraically closed field \bar{k} , the subset of unipotent elements \bar{L}_u is a (closed) connected, normal \bar{k} -subgroup, and the quotient is an affine torus.* In the situation of the Néron model, this torus can be obtained by pull-back from a torus over $\kappa(s)$. Thus *the (subgroup $X_{S,s}^\circ$ of the) Néron model is a semi-abelian scheme, if and only if \bar{L}_u is trivial.*

Remark 3.9. The standard way to assert that \bar{L}_u is trivial is to assert that the unipotent radical of \bar{L} is trivial. Indeed, for a connected, solvable group G over an algebraically closed field, the radical $\mathcal{R}G$ is equal to the group G (the radical is the largest connected, solvable, normal subgroup; e.g. [26, p.157]). Thus in this case the unipotent radical $(\mathcal{R}G)_u =: \mathcal{R}_u G$ (the set of unipotent elements of the radical) is equal to the set G_u .

3.4. The stable reduction theorem. The stable reduction theorem plays a central role in the study of abelian varieties, and also in the study of algebraic curves. In light of the results of Néron, and the basic structure theorems for algebraic groups, the stable reduction theorem states that after a finite base change, the unipotent radical of the central fiber of the Néron model can be made trivial.

As described in the introduction to [44], the stable reduction theorem was first proved independently by Grothendieck and Mumford in characteristic zero. Grothendieck's proof used the theory of étale cohomology, while Mumford's proof was derived from a stable reduction theorem for curves (in characteristic zero). Grothendieck then extended his proof to all characteristics in [97, Thm. 6.1, p.21] and Mumford provided an independent proof in characteristics other than 2 using the theory of theta functions.

Theorem 3.10 (Grothendieck–Mumford Stable Reduction Theorem). *Let $S = \operatorname{Spec} R$ be the spectrum of a DVR with fraction field K . An abelian variety A_K over K has potential semi-abelian reduction over R .*

We refer the reader also to [27, Thm. 1, p.180]. Grothendieck’s proof relies on the following frequently cited result, which was the basis of the monodromy obstruction computation in §1.

Proposition 3.11 (Grothendieck [97, Prop. 3.5, p.350]). *In the notation above, A_K has abelian (resp. semi-abelian) reduction if and only if the monodromy representation is trivial (resp. unipotent).*

We direct the reader also to [27, Thm. 5, p.183]. The Grothendieck–Mumford Stable Reduction Theorem follows from the proposition and the Monodromy Theorem after the observation (see Remark 2.2) that quasi-unipotent monodromy can be made unipotent after a finite base change.

3.5. Alexeev’s space of stable semiabelic pairs. One would like to derive from the Grothendieck–Mumford Stable Reduction Theorem a properness statement for a moduli space. This serves as motivation to introduce Alexeev’s compactification [8] of the moduli space of principally polarized abelian varieties, where such a statement holds.

Let us recall some definitions from [8] (we also direct the reader to Olsson [93] for a related moduli problem). First a reduced scheme X is said to be **semi-normal** if given any proper, bijective morphism $f : X' \rightarrow X$ from a reduced scheme X' satisfying the property that $\kappa(f(x')) \rightarrow \kappa(x')$ is an isomorphism for all $x' \in X$, then f is an isomorphism (see e.g. [74, §7.2], [8, 1.1.6]). For instance, a nodal curve is semi-normal, while a cuspidal curve is not.

A **stable semiabelic variety** ([8, 1.1.5]) is a semi-normal, equidimensional, reduced scheme X over an algebraically closed field k , together with an action of a connected semi-abelian scheme G/k of the same dimension, such that there are only finitely many orbits for the G -action, and the stabilizer group scheme of every point of X is connected, reduced and lies in the toric part of G .

A **polarized stable semiabelic variety** ([8, 1.1.8]) is a projective stable semiabelic variety together with an ample invertible sheaf L . The degree of the polarization is defined as $h^0(L)$. A **stable semiabelic pair** (X, Θ) consists of a polarized stable semiabelic variety X with ample invertible sheaf L together with a section $\theta \in H^0(X, L)$ that does not vanish on any G -orbits. So in total, for a stable semiabelic pair, we have the data (X, G, L, θ) . We take Θ to be the zero set of θ , and use the shorter notation (X, Θ) to indicate the connection to polarized abelian varieties.

We now make the relative definition. For a scheme B , a **stable semiabelic pair over B** , denoted (X_B, Θ_B) , is the data

$$(X_B, G_B, L_B, \theta_B)$$

where $\pi_B : X_B \rightarrow B$ is a projective, flat morphism, G_B is a semi-abelian scheme over B acting on X_B , L_B is a relatively ample line bundle on X_B , $\theta_B \in H^0(B, \pi_* L_B)$, and the restriction of this data to every geometric point $\bar{b} \rightarrow B$ is a stable semiabelic pair over \bar{b} . ([8, p.617]). One can show that $\pi_* L_B$ is locally free and that this push forward commutes with arbitrary base change. The degree is defined to be the rank of $\pi_* L_B$.

For brevity, we do not give a precise definition of Alexeev's stack $\bar{\mathcal{A}}_g^A$, and say only that it is a substack of the stack of all stable semiabelic pairs of rank 1 and dimension g . The stack $\bar{\mathcal{A}}_g^A$ contains a component that has \mathcal{A}_g , the moduli stack of principally polarized abelian varieties of dimension g , as a dense open substack. Alexeev proves [8, Thm 5.10.1] that $\bar{\mathcal{A}}_g^A$ is a proper, algebraic (Artin) stack over \mathbb{Z} with finite diagonal. Moreover, the stack admits a coarse moduli space, with a component that has normalization isomorphic to the second voronoi compactification \bar{A}_g^{Vor} [8, Thm. 5.11.6, p. 701]. To establish properness, Alexeev proves the following stable reduction theorem for semiabelic pairs.

Theorem 3.12 (Alexeev [8, Thm. 5.7.1, p.692]). *Let $S = \text{Spec } R$ be the spectrum of a DVR with fraction field K . Let (X_K, Θ_K) be a stable semiabelic pair over K . Then there is a DVR R' with field of fractions K' , a finite morphism $S' = \text{Spec } R' \rightarrow S$ and a stable semiabelic pair $(X_{S'}, \Theta_{S'})$ extending the pull-back $(X_{K'}, \Theta_{K'})$. Moreover, the central fiber $(X_{s'}, \Theta_{s'})$ of $(X_{S'}, \Theta_{S'})$ is unique up to isomorphism.*

Example 3.13. Assume the residue field of R is algebraically closed. Consider a family $C \rightarrow S$ of smooth curves degenerating to an irreducible, stable curve C_s with a single node. Let C_s^ν be the normalization of C_s , and assume that C_s is obtained from C_s^ν by attaching points $p, q \in C_s^\nu$. The family of curves C_K over K determines a principally polarized abelian scheme (X_K, Θ_K) , where $X_K = JC_K$ is the Jacobian of the curve. Let $(X_{S'}, \Theta_{S'})$ be a stable reduction of X_K .

The central fiber can be described as follows. The degree zero Picard functor applied to C_S determines a semi-abelian scheme over S . The central fiber is the semi-abelian scheme

$$0 \rightarrow \mathbb{G}_m \rightarrow G \rightarrow JC_s^\nu \rightarrow 0$$

determined by the data of the line bundle $\mathcal{O}_{C_s^\nu}(p - q)$. The group G can be completed to a \mathbb{P}^1 -bundle over JC_s^ν , with sections σ_0 and σ_∞ . The fiber $X_{s'}$ is obtained from this projective bundle by gluing the sections transversally, after shifting by $\mathcal{O}_{C_s^\nu}(p - q)$. Note that G acts on this space. We direct the reader to [9] for more details and a description of the polarization (see also [87]).

3.6. Simultaneous stable reduction for abelian varieties. We now consider the case of extending families of abelian varieties over bases other than a DVR. The main result we mention is due to Faltings–Chai [47].

Theorem 3.14 (Faltings–Chai Extension [47, Thm. 6.7, p.185]). *Let B be a regular scheme over a field of characteristic 0. Let $\Delta \subseteq B$ be an nc divisor. Let A_U be an abelian scheme over $U = B - \Delta$, which extends to a semi-abelian scheme A_V over an open subscheme V containing U and the generic points of Δ . Then A_U extends uniquely to a semi-abelian scheme A_B over B .*

Remark 3.15. This fails in positive characteristic. A counter example when the characteristic of the generic points of B are positive is given in [47, p.192]. A counter example of Raynaud–Ogus–Gabber, when the characteristic of the generic points of B are zero (but where other points have positive characteristic), is given in de Jong–Oort [43, §6].

From the Faltings–Chai Extension Theorem, one obtains a purity lemma.

Corollary 3.16 ([47, Cor. 6.8, p.185]). *If B is a regular scheme over a field of characteristic 0, and $Z \subseteq B$ is a closed subscheme of codimension ≥ 2 , then any abelian scheme over $U = B - Z$ extends uniquely to an abelian scheme over B .*

The proof is essentially to reduce to the case of a point in a surface, and then to use regularity to exhibit the point as a transverse intersection of curves, at which point one can apply the theorem.

The Faltings–Chai theorem also implies a special case of the Borel Extension Theorem. Recall that we use the notation \mathcal{A}_g for the stack of principally polarized abelian varieties of dimension g . A morphism $U \rightarrow \mathcal{A}_g$ corresponds to a family $A_U \rightarrow U$ of principally polarized abelian varieties. We denote the coarse moduli space by A_g . We denote by A_g^* the Satake (Bailey–Borel) compactification, and by \bar{A}_g any one of Mumford’s toroidal compactifications. The most common toroidal compactification we will use is the second Voronoi, which we will denote by \bar{A}_g^{Vor} . We direct the reader to [90] for more details.

Theorem 3.17 (Borel Extension [25, Thm. A]). *Let B be a regular scheme over a field of characteristic 0. Let $\Delta \subseteq B$ be an nc divisor. Setting $U = B - \Delta$, then for any morphism $f : U \rightarrow \mathcal{A}_g$, the composition $U \rightarrow \mathcal{A}_g \rightarrow A_g$ extends to a morphism $B \rightarrow A_g^*$.*

Borel’s proof uses hyperbolic complex analysis and holds more generally for (locally liftable) holomorphic maps into Bailey–Borel compactifications of arithmetic quotients of bounded symmetric domains. Faltings–Chai [47, Cor. 6.11, p.191] also prove the related statement for maps into the moduli space $A_g[n]$ of principally polarized abelian varieties with level n -structure for $n \geq 3$. In this case they can use [88, Thm. 7.9, Thm. 7.10, p.139] to conclude that the coarse moduli space is quasi-projective and fine. In other words, in both this situation, as well as under the assumptions of the Borel extension theorem as stated above, one may assume there is a family of abelian varieties over U .

The argument from there is short. First, the extension statement is local. One can also show that it suffices to establish extension after a finite base change (e.g. [36, Lem. 2.4]). Thus we may take an étale base change, and assume we are in the situation where B is regular and Δ has support defined by $x_1 \cdots x_r$, where x_1, \dots, x_r form part of a system of local parameters. Taking the finite cover $t_1 = x_1^{m_1}, \dots, t_r = x_r^{m_r}$ for appropriate values of m_1, \dots, m_r , one uses the monodromy theorem to get extension over the generic points of Δ . The result then follows from the Faltings–Chai Extension Theorem.

Remark 3.18. The condition in the Borel Extension Theorem that there is a family of abelian varieties over U (or more generally that the holomorphic map is locally liftable to the bounded symmetric domain) is essential. More precisely, for B and U as in the theorem, given a morphism $f : U \rightarrow A_g$, this need not extend to a morphism $B \rightarrow A_g^*$. An elementary example comes from the case of A_1 and A_1^* . We can identify A_1 as the quotient $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ of the upper half plane by the special linear group in the usual way, and it is well known that A_1^* is isomorphic to $\mathbb{P}_{\mathbb{C}}^1$. The map $(\mathbb{C}^*)^2 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ given by $(\lambda_1, \lambda_2) \mapsto [\lambda_1 : \lambda_2]$ clearly does not extend to a morphism from \mathbb{C}^2 .

Remark 3.19. It is natural to ask whether a statement like the Borel extension theorem could hold for the toroidal compactifications of A_g , and indeed there is an extension theorem due to Ash–Mumford–Rapaport–Tai [19] (see also Namikawa [90, Thm. 7.29, p.78]) giving explicit conditions for morphisms to extend over nc boundaries. In concrete examples these extension conditions can be difficult to establish. We discuss some particular examples below.

The previous remark concerns extension results for maps to the toroidal compactifications. For the case of the second Voronoi compactification, one could also ask for extensions of morphisms to Alexeev’s moduli stack of semi-abelic pairs. We expect the following result is well-known to the experts, but we were not aware of a reference. We provide a proof in the next section, using an observation of Fedorchuk and the fact that the stack is proper with finite diagonal.

Theorem 3.20. *A rational map $B \dashrightarrow \bar{A}_g^A$ from a quasi-compact, quasi-separated (or Noetherian) scheme B is resolved by an alteration.*

3.6.1. *Examples of period maps to A_g .* Having made the connection between period maps into the moduli of abelian varieties and the problem of simultaneous semi-abelian reduction, we now consider a few well known examples of period maps. In this section we will work over \mathbb{C} . The most well known example is the Torelli map for curves; i.e. the morphism

$$\mathcal{T} : \mathcal{M}_g \rightarrow A_g$$

that sends a curve C to its principally polarized Jacobian (JC, Θ_C) . Let $T : M_g \rightarrow A_g$ be the associated morphism of coarse moduli spaces. Torelli’s theorem states that T is injective.

The boundary Δ in \overline{M}_g is (up to finite quotient singularities) an nc divisor. As a consequence of the Borel extension theorem, we obtain a morphism

$$T^* : \overline{M}_g \rightarrow A_g^*$$

extending T . For the toroidal compactifications of A_g , there are the general extension results mentioned above. In practice, these can be difficult to verify. It is a result of Mumford and Namikawa [90], [89, §18] that T extends to a morphism

$$\overline{T}^{Vor} : \overline{M}_g \rightarrow \overline{A}_g^{Vor}.$$

Caporaso–Viviani describe the fibers of the morphism in [30]. In addition, it is shown in Alexeev [9] that there is a morphism $\overline{\mathcal{T}}^{Vor} : \overline{M}_g \rightarrow \overline{\mathcal{A}}_g^A$ extending \mathcal{T} . We direct the reader to Alexeev–Brunyate [11] for a proof that the Torelli map for stable curves extends to a morphism to the first Voronoi compactification (see also Gibney [55] for more on the image of the Torelli map to other toroidal compactifications).

We now turn our attention to the Prym map. We denote by \mathcal{R}_g the moduli stack of connected, étale double covers of non-singular curves of genus g . The Prym map

$$Pr : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$$

takes a double cover $\pi : \tilde{C} \rightarrow C$ to its principally polarized Prym variety (P, Ξ) (see Mumford [85] for more details). We denote by $Pr : \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$ the associated morphism of coarse moduli spaces. It is well known that the map is dominant for $g \leq 6$ (see esp. [22]), and in the other direction, Friedman–Smith [52] and Kanev [67] have shown that the map is generically injective for $g \geq 7$. There is a compactification, $\overline{\mathcal{R}}_g$, due to Beauville [22], consisting of admissible double covers. The coarse moduli space $\overline{\mathcal{R}}_g$ has (up to finite quotient singularities) an nc boundary. As a consequence, there is an extension

$$Pr^* : \overline{\mathcal{R}}_g \rightarrow A_{g-1}^*.$$

On the other hand, Friedman–Smith [53] have shown that the Prym map does not extend to a morphism to any of the Toroidal compactifications. We direct the reader to Alexeev–Birkenhake–Hulek [10] for more details on the indeterminacy locus of the Prym map to \overline{A}_{g-1}^{Vor} .

The Clemens–Griffiths [38] period map for cubic threefolds provides another interesting example. Recall that a cubic threefold is a smooth cubic hypersurface $X \subseteq \mathbb{P}^4$. The intermediate Jacobian of X is the five dimensional complex torus $JX := H^{1,2}(X)/H^3(X, \mathbb{Z})$. This admits a principal polarization Θ_X , given by the hermitian form h on $H^{1,2}(X)$ defined by $h(\alpha, \beta) = 2i \int_X \alpha \wedge \bar{\beta}$. Letting M_{cub} be the moduli space of cubic threefolds, one obtains a morphism

$$J : M_{cub} \rightarrow A_5.$$

By virtue of the Clemens and Griffiths Torelli theorem [38] (see also Mumford [85]), J is injective. We denote the image by I , and we direct the reader

to Casalaina-Martin–Friedman [33] for a geometric characterization of the abelian varieties parameterized by I .

The space M_{cub} admits a GIT compactification

$$\overline{M}_{cub} = \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) // \mathrm{SL}(5),$$

(see Allcock [12], Yokoyama [104]) and it is natural to consider extensions of the period map J to \overline{M}_{cub} . Allcock–Carlson–Toledo [13] and Looijenga–Swierstra [81] have shown that M_{cub} can be identified with an open dense subset of a ten dimensional ball quotient \mathcal{B}/Γ . They show moreover, that the rational map $\overline{M}_{cub} \dashrightarrow (\mathcal{B}/\Gamma)^*$ to the Baily–Borel compactification, can be resolved by blowing up a single point. We call the resulting space \widehat{M}_{cub} .

Using the description of \widehat{M}_{cub} given in [13, 81], Laza and the author describe an explicit blow-up \widetilde{M}_{cub} of \widehat{M}_{cub} , with discriminant an nc divisor [34]. The process used to obtain the resolution is the same as that described for simultaneous stable reduction for ADE curves below (see §4.4). The Borel extension theorem then gives a morphism

$$J^* : \widetilde{M}_{cub} \rightarrow A_5^*.$$

Laza and the author use this extension of the period map together with results from [85, 22, 33, 32] and the explicit description of \widetilde{M}_{cub} to describe the boundary of the image of J^* [34, Thm. 1.1].

Remark 3.21. An explicit resolution of the map $\overline{M}_{cub} \dashrightarrow \overline{A}_5^{Vor}$ is still not known. Certain components of the boundary of the (closure of the) image have been identified by Grushevsky–Salvati Manni [60] and Grushevsky–Hulek [59] via the theory of theta functions.

4. CURVES

In this section, we start by describing the connection between stable reduction for curves and stable reduction for abelian varieties. We then discuss some recent progress on the problem of simultaneous stable reduction for curves.

4.1. Stable reduction for curves. Recall that a stable curve X over an algebraically closed field is a pure dimension 1, reduced, connected, complete scheme of finite type, with at worst nodes as singularities, and with finite automorphism group. The genus is defined as $g = h^1(X, \mathcal{O}_X)$. For a scheme S , a stable curve X/S is a proper, flat morphism $X \rightarrow S$ whose geometric fibers are stable curves.

Theorem 4.1 (Deligne–Mumford Stable Reduction [44]). *Let $S = \mathrm{Spec} R$ be the spectrum of a DVR with fraction field K . Let X_K be a stable curve over K . Then there is a DVR R' with fraction field K' , a finite morphism $S' = \mathrm{Spec} R' \rightarrow S$ and a stable curve $X_{S'}$ over S' extending the pull-back $X_{K'}$. Moreover, the central fiber $X_{s'}$ of $X_{S'}$ is unique up to isomorphism.*

In characteristic zero, there is a direct proof due to Mumford using a special case of the Semi-stable Reduction Theorem, which we will sketch in §5. In positive characteristic, the proof in [44] is made via the stable reduction theorem for abelian varieties. The outline of this proof is as follows. Take X_K smooth for simplicity and consider the Jacobian J_K/K . The stable reduction theorem implies this extends to a family of semi-abelian varieties, at least after a finite base change. It is then shown:

Theorem 4.2 (Deligne–Mumford [44, Thm. 2.4]). *A family of stable curves X_K/K extends to a family of stable curves over S if and only if the associated family of Jacobians J_K/K has semi-abelian reduction over S .*

The proof uses the result of Raynaud’s mentioned in Remark 3.7. It is also worth mentioning a converse argument: the stable reduction theorem for curves implies the stable reduction theorem for abelian varieties. We direct the reader to [27, p.182] for a more detailed discussion. The idea is to first use the previous theorem that stable reduction for curves implies stable reduction for Jacobians. An abelian scheme A_K can be viewed as the quotient of a product of Jacobians; i.e.

$$0 \rightarrow A'_K \rightarrow J_K \rightarrow A_K \rightarrow 0$$

where A'_K is an abelian scheme, and J_K is a product of Jacobians (e.g. Serre [96, Cor. p.180]). One then shows that in general, for such an extension of abelian schemes, J_K has semi-abelian reduction if and only if A_K and A'_K do, completing the proof. Finally, we note that Artin–Winters have given another proof of stable reduction for curves in positive characteristic, which does not rely on the stable reduction theorem for abelian varieties [18].

Remark 4.3. While unipotent monodromy for a one-parameter family of smooth curves implies the family extends to a family of stable curves, trivial monodromy does not necessarily imply that the central fiber of the extension is smooth. For instance, a one-parameter family of smooth curves degenerating to a singular, stable curve of compact type will have associated Jacobian that is an abelian scheme over the base. Grothendieck’s theorem implies that the monodromy of the family will be trivial. We direct the reader to Oda [92, Thm. 10] for a statement concerning a related monodromy invariant that detects when a one-parameter family can be extended to a smooth curve.

Remark 4.4. There is a stable reduction theorem for pointed curves as well. Pointed curves provide a natural introduction to the important topic of moduli spaces of pairs. For the sake of brevity, we have generally avoided this topic in the presentation here. It will, however, be of central importance in §5.3 on slc models, and it is worth noting this example as a precursor.

4.2. Simultaneous stable reduction for curves. In analogy with the Faltings–Chai extension theorem, we mention the following simultaneous stable reduction theorem of de Jong–Oort [43].

Theorem 4.5 (de Jong–Oort Extension [43, Thm. 5.1]). *Let B be a regular scheme and Δ an nc divisor on B . Set $U = B - \Delta$. A family of smooth curves of genus $g \geq 2$ over U extends to a family of stable curves over B if it extends to a family of stable curves over an open subset V containing each generic point of Δ .*

In fact the theorem is more general, in that one can allow for a generically stable family, so long as the topological type is locally constant on U . A similar result was proven by Moret-Bailly [83], where it is required that a generically smooth family extend to a *smooth* family over the generic points of Δ .

A consequence of the de Jong–Oort Extension Theorem is an analogue of the Borel Extension Theorem for stable curves. Before stating the result, let us first rephrase the previous theorem in the language of stacks. The theorem states that given a morphism to the stack $U \rightarrow \mathcal{M}_g$, there is an extension $B \rightarrow \overline{\mathcal{M}}_g$ if and only if there is an open set $V \subseteq B$ containing U and the generic points of Δ and an extension $V \rightarrow \overline{\mathcal{M}}_g$.

Corollary 4.6 (Cautis [36, Thm. A]). *Let B be a regular scheme and Δ an nc divisor on B . Set $U = B - \Delta$. Given a morphism $U \rightarrow \mathcal{M}_g$, there is an extension $B \rightarrow \overline{\mathcal{M}}_g$.*

One obtains this corollary from the previous theorem in the same way as the analogous statement was proven for semi-abelian varieties (i.e. in the way the Borel Extension Theorem follows from the Faltings–Chai Extension Theorem).

An independent proof of the corollary was given by Cautis [36, Thm. A]. By virtue of $\overline{\mathcal{M}}_g$ being a separated Deligne–Mumford stack, it is immediate to prove the de Jong–Oort Extension Theorem from the corollary using the Abramovich–Vistoli purity lemma [4, Lemma 2.4.1].

4.3. Simultaneous stable reduction by alterations. We now turn our attention to the following related problem: *given a scheme B and a rational map $B \dashrightarrow \overline{\mathcal{M}}_g$ determined by a morphism $U \rightarrow \overline{\mathcal{M}}_g$ for some open subset $U \subseteq B$, find a resolution of the rational map.*

If the base of the family is allowed to be a stack, then the solution is clear: resolve the induced rational map $B \dashrightarrow \overline{\mathcal{M}}_g$ by taking the closure of the graph of U in $B \times_{\mathbb{Z}} \overline{\mathcal{M}}_g$ to obtain $\tilde{B} \rightarrow \overline{\mathcal{M}}_g$. Then take the fibered product $\tilde{B} \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_g$. On the other hand, it may be preferable in many instances to remain in the category of schemes. We mention now a theorem of de Jong. The following is a special case, which will be general enough for our discussion here.

Theorem 4.7 (de Jong [42, Thm. 5.8]). *Given a quasi-compact, quasi-separated (or Noetherian) scheme B and a rational map $B \dashrightarrow \overline{\mathcal{M}}_g$, there exists an alteration $\tilde{B} \rightarrow B$ resolving the rational map.*

For this special case of the theorem, there is another argument, pointed out to the author by Fedorchuk (see [50, Rem. 7.3]), which follows from [46, Thm. 2.7]. The basic observation has implications beyond just the moduli of curves, and we present the argument here. The first point is the following lemma on algebraic stacks.

Lemma 4.8 (Fedorchuk [50, Rem. 7.3]). *Let S be a noetherian scheme. Let \mathcal{M} be an algebraic (Artin) S -stack, proper over S , that admits a finite, surjective morphism*

$$V \rightarrow \mathcal{M}$$

from a scheme V . Then any rational map $B \dashrightarrow \mathcal{M}$ from a quasi-compact, quasi-separated (or Noetherian) S -scheme B can be resolved by an alteration.

Proof. The proof (following Fedorchuk [50]) is short and we include it here. Consider the finite morphism $V \rightarrow \mathcal{M}$ assumed in the statement of the lemma. Note we obtain that V is proper over S , since V is finite (and hence proper) over \mathcal{M} and we have assumed that \mathcal{M} is proper over S .

Let $B^\circ \rightarrow \mathcal{M}$ be the morphism inducing the rational map $B \dashrightarrow \mathcal{M}$. From the definition of an algebraic stack, the diagonal is representable. Consequently, $B^\circ \times_{\mathcal{M}} V$ is a scheme. We then have a commutative diagram

$$(4.1) \quad \begin{array}{ccc} B^\circ \times_{\mathcal{M}} V & \longrightarrow & V \\ \downarrow \text{finite} & & \downarrow \text{finite} \\ B^\circ & \longrightarrow & \mathcal{M}. \end{array}$$

Let $B' \rightarrow B$ be a finite morphism extending $B^\circ \times_{\mathcal{M}} V \rightarrow B^\circ$; to obtain this extension one can either use Zariski's Main Theorem [58, EGA IV.3 Thm. 8.12.6, p.45] or [48, Lem. 5.19, p.131] (in the latter case, one extends the push forward of the structure sheaf of $B^\circ \times_{\mathcal{M}} V$ to a coherent sheaf on B and then takes the relative spectrum). We thus obtain a rational map $B' \dashrightarrow V$.

Let \tilde{B} be the closure of the graph of $B^\circ \times_{\mathcal{M}} V \rightarrow V$ in $B' \times_S V$. The morphism $B' \times_S V \rightarrow B'$ is proper by base change, and a closed immersion is proper. It follows that $\tilde{B} \rightarrow B'$ is proper, and also birational by construction. Thus the composition

$$\tilde{B} \rightarrow B' \rightarrow B$$

gives an alteration that resolves the map to \mathcal{M} . \square

Remark 4.9. The assumption that the scheme B be quasi-compact and quasi-separated, or that it be Noetherian, was used to ensure the existence of a finite cover of B extending the given finite cover of B° . Another approach could be to assume that B is covered by the spectrums of Japanese rings. In this case, the appropriate integral closures will be finitely generated, allowing for another construction of a finite cover.

The utility of the lemma comes from the following theorem of Edidin–Hassett–Kresch–Vistoli [46].

Theorem 4.10 (Edidin et al. [46, Thm. 2.7]). *Suppose S is a noetherian scheme. Let \mathcal{M} be an algebraic (Artin) S -stack of finite type over S . Then the diagonal*

$$\mathcal{M} \rightarrow \mathcal{M} \times_S \mathcal{M}$$

is quasi-finite if and only if there exists a finite, surjective morphism $V \rightarrow \mathcal{M}$ from a (not necessarily separated) scheme V .

Remark 4.11. Recall that the diagonal morphism of an algebraic stack is quasi-compact by assumption, and an algebraic stack is Deligne–Mumford if and only if the diagonal is unramified. A quasi-compact, unramified morphism of schemes is quasi-finite. In other words, Deligne–Mumford stacks have quasifinite diagonal (see e.g. [46, Rem. 2.5] for a converse in characteristic 0). We also direct the reader to [80, Thm. 6.2] for a related, well known statement on the local structure of Deligne–Mumford stacks.

The proof of the special case of de Jong’s theorem mentioned above (Theorem 4.7) is then a direct consequence of the theorem of Edidin et al. and the lemma of Fedorchuk (Theorem 4.10 and Lemma 4.8). We can now also provide a proof of Theorem 3.20.

Proof of Theorem 3.20. Alexeev has shown that the stack $\bar{\mathcal{A}}_g^A$ is a proper algebraic (Artin) stack over \mathbb{Z} with finite diagonal; thus Theorem 3.20 is a direct consequence of the results mentioned above. \square

4.4. Explicit simultaneous stable reduction for curves. Having established the existence of alterations resolving rational maps to $\bar{\mathcal{M}}_g$, one can ask for explicit alterations in specific settings. One place where this type of question arises naturally is in the Hassett–Keel program for the moduli space of curves.

We will not discuss the details of the Hassett–Keel program here, but will simply note that in this program, projective varieties $\bar{\mathcal{M}}_g(\alpha)$, $\alpha \in [0, 1] \cap \mathbb{Q}$ arise, which are conjectured to parameterize curves of genus g with prescribed singularities (for $0 \ll \alpha \leq 1$ this has been established in Hassett–Hyeon [65, 64]). For “most” g and α there are birational maps

$$\bar{\mathcal{M}}_g(\alpha) \dashrightarrow \bar{\mathcal{M}}_g$$

to the moduli space of stable curves. It would be of interest to have explicit resolutions. In general, these birational maps will lift to rational maps to the stack $\bar{\mathcal{M}}_g(\alpha) \dashrightarrow \bar{\mathcal{M}}_g$, and in this way we arrive at the related problem of simultaneous stable reduction.

With this as motivation, we will consider the following problem. *Given a generically smooth family of curves $X \rightarrow B$ with fibers having prescribed singularities, give an explicit description of a resolution of the rational map $B \dashrightarrow \bar{\mathcal{M}}_g$.*

4.4.1. *Simultaneous stable reduction for curves with ADE singularities.* The specific case we will consider is where the singular fibers have at worst *ADE* singularities (we review the definition of *ADE* singularities in §6). We call such curves *ADE* curves, and we will consider the question (étale) locally.

Laza and the author have given a solution to this problem in [35] and Fedorchuk has given an independent solution for singularities of type *AD* in [50]. Fedorchuk's proof is based on constructions of proper moduli spaces of hyperelliptic curves $\mathcal{H}[k, \ell]$, where the boundary consists of certain curves with *AD* singularities at worst of type A_k and D_ℓ . The proof provides modular descriptions of the spaces arising in the processes described below. We direct the reader to Fedorchuk [50] for more details.

Below is the version of the result in [35]. Since we consider the resolution question (étale) locally, it suffices to understand the case where $X \rightarrow B$ is a mini-versal deformation of an *ADE* curve X_0 . The statement of the theorem uses the notion of a Weyl cover, and wonderful blow-up; these are explicit maps discussed further below, which can be determined by the root systems associated to the singularities. For the statement of the theorem, we note that the wonderful blow-up of the Weyl cover of B has the property that the pull-back of the discriminant is an nc divisor, with irreducible components corresponding to curves with fixed singularity type.

Theorem 4.12 (Casalaina-Martin-Laza [35], Fedorchuk [50]). *Let $X \rightarrow B$ be a mini-versal deformation of an ADE curve X_0 with $p_a(X_0) = g \geq 2$. The wonderful blow-up of the Weyl cover of B resolves the rational moduli map to the moduli scheme \overline{M}_g , but fails to resolve the rational moduli map to the moduli stack $\overline{\mathcal{M}}_g$ along the A_{2n} locus of the discriminant ($n \in \mathbb{N}$). The addition of a stack structure (generically $\mathbb{Z}/2\mathbb{Z}$ stabilizers) along this locus resolves the moduli map to $\overline{\mathcal{M}}_g$.*

Remark 4.13. Let us elaborate on the final statement in the theorem concerning stacks. There is a family of stable curves over the wonderful blow-up of the Weyl cover, except over the locus parameterizing curves with A_{2n} singularities. This locus is a collection of divisors, and there is an obstruction to extending the family over that locus. At the generic points, the obstruction becomes trivial after taking a branched double cover.

We now outline the proof in [35], which can be viewed as a generalization of the steps taken in the example in §1. This also provides some description of the Weyl cover and the wonderful blow-up. The first observation is that the base B splits, up to a smooth factor, as a product of the miniversal spaces of the singularities of the curve X_0 (see e.g. [35, §1] for a review of this). Returning to the example from §1 for illustration, the family can also be viewed as a miniversal deformation of a cusp (an A_2 singularity).

The Weyl cover. Miniversal deformations of *ADE* singularities are well understood. In particular, there is the notion of a Weyl cover, which gives a finite morphism $B' \rightarrow B$, such that after pull-back, the discriminant is an

arrangement of hyperplanes determined by the product of the root systems associated to the singularities (see §6.2.1 and also [35, §2] for a review of this). Returning to the example from §1 for illustration, the Weyl cover is the finite Σ_3 cover described in §1.2.1 and the pull-back of the discriminant is the hyperplane arrangement associated to an A_2 root system (corresponding to the A_2 singularity).

The wonderful blow-up. With the discriminant given by a hyperplane arrangement associated to a root system, there is a theory of wonderful blow-ups due de Concini–Procesi [41] (see also [35, §3]). Roughly speaking, one blows-up the arrangement of hyperplanes inductively starting with the highest codimension strata to obtain a smooth space with discriminant in normal crossing position. We call the space obtained $\tilde{B} \rightarrow B'$ the wonderful blow-up of the Weyl cover (it is well defined up to isomorphism). In the example in the first section, this corresponds to the blow-up described in §1.2.2.

Completing the proof. With the discriminant in nc position, it follows from the de Jong–Oort extension theorem (see Corollary 4.6) that there is a morphism $\tilde{B} \rightarrow \overline{M}_g$. The remaining statements of Theorem 4.12 can be obtained by showing the monodromy is unipotent except around the generic point of an A_{2n} stratum of the discriminant, where it is not unipotent, but its square is (see [35, Cor. 5.4]). One can also make a direct computation in coordinates, as in §1, and Fedorchuk [50] indicates another method via a construction with hyperelliptic curves.

Remark 4.14. As mentioned above in the section on period maps to the moduli space of abelian varieties, the method of proof of this theorem has applications to other situations including the study of the moduli space of cubic threefolds [34].

5. (SEMI-)STABLE REDUCTION FOR HIGHER DIMENSIONAL VARIETIES

We now consider stable reduction problems for families of varieties of higher dimension. There are two main approaches we will follow in this setting. The first approach, which we will call semi-stable reduction, aims to replace the central fiber of a family of varieties (after an appropriate base change, and modification of the total space) with a simpler, but possibly non-unique scheme. Specifically, in the case when the base of the family is dimension 1, the aim will be to replace the central fiber of the family with a scheme that is a reduced, normal crossings divisor in the total space. This is the natural generalization of filling in the central fiber in a family of smooth curves with a nodal curve.

The second approach to the problem is to define classes of schemes so that one-parameter families of schemes in this class can be filled in (possibly after a finite base change) with a unique (up to isomorphism) scheme in this class. This is the analogue of the Deligne–Mumford Stable Reduction Theorem, and is in most settings a very difficult problem. We discuss some recent

results on varieties of general type due to Kollár [75, 76], which generalize results of Kollár–Shepherd-Barron [78] and Alexeev [5, 6, 7] for surfaces of general type.

5.1. Semi-stable reduction. We start with the question of semi-stable reduction. We will consider the case due to Mumford et al. [71] for one-parameter families, and then consider some recent progress due to de Jong [42] and Abramovich–Karu [3] on the case of higher dimensional bases.

5.1.1. Semi-stable reduction for one-parameter families. For one-parameter families there is the well known **Semi-stable Reduction Theorem** of Kempf–Knudsen–Mumford–Saint-Donat [71].

Theorem 5.1 (Semi-stable Reduction Theorem [71, Thm. p.53]). *Assume that $\text{char}(k) = 0$ and $k = \bar{k}$. Let B be an open subset of a non-singular curve over k , fix a point $o \in B$, and set $U = B - \{o\}$. Suppose that*

$$\pi : X \rightarrow B$$

is a surjective morphism of a variety X onto B such that the restriction $\pi_U : X|_U \rightarrow U$ is smooth. Then there is a finite base change $f : B' \rightarrow B$, with B' non-singular and $f^{-1}(o)$ a single point o' , a non-singular variety X' and a diagram

$$(5.1) \quad \begin{array}{ccccc} X' & \xrightarrow{\quad p \quad} & B' \times_B X & \xrightarrow{\quad \quad} & X \\ & \searrow \pi' & \downarrow & & \downarrow \pi \\ & & B' & \xrightarrow{\quad f \quad} & B \end{array}$$

satisfying the properties below.

- (1) Setting $U' = B' - \{o'\}$, p is an isomorphism over U' .
- (2) $(\pi')^{-1}(o')$ is a reduced scheme, which is an snc divisor on X' .
- (3) The morphism p is projective, and given as a blow-up of an ideal sheaf \mathcal{I} that is trivial away from the fiber over o' .

This result is used so frequently in stable reduction arguments, and parts of the proof are so constructive, that it is worthwhile to sketch the outline here. One of the key points is the following example.

Example 5.2. Consider the variety X in $\text{Spec } k[\underline{x}, t] = \mathbb{A}_k^{r+1}$ defined by

$$t - x_1^{a_1} \cdots x_r^{a_r}.$$

We view X as a family $\pi : X \rightarrow B := \text{Spec } k[t]$, with central fiber $D = \pi^{-1}(0)$. For each $d \in \mathbb{N}$, set $B_d = \text{Spec } k[t]$, and $f_d : B_d \rightarrow B$ to be the map given by $t \mapsto t^d$. We define X_d to be the normalization of the pull-back of X via the map $B_d \rightarrow B$. In other words, we have a diagram

$$\begin{array}{ccccc} X_d & \xrightarrow{\quad \nu_d \quad} & B_d \times_B X & \xrightarrow{\quad \quad} & X \\ & \searrow \pi_d & \downarrow & & \downarrow \pi \\ & & B_d & \xrightarrow{\quad f_d \quad} & B \end{array}$$

In this example, we will assume that

$$d = \text{lcm}(a_1, \dots, a_r) \quad \text{and} \quad \gcd(d, a_1, \dots, a_r) = \gcd(a_1, \dots, a_r) = 1,$$

and we will describe X_d and $\pi_d^{-1}(0)$. First, $X_{B_d} := B_d \times_B X$ is defined by

$$t^d - x_1^{a_1} \cdots x_r^{a_r}.$$

The assumption $\gcd(d, a_1, \dots, a_r) = \gcd(a_1, \dots, a_r) = 1$ implies that X_{B_d} is the image of the morphism

$$(5.2) \quad \text{Spec } k[\underline{y}] = \mathbb{A}_k^r \rightarrow \mathbb{A}_k^{r+1} = \text{Spec } k[\underline{x}, t]$$

given by $(y_1, \dots, y_r) \mapsto (y_1^d, \dots, y_r^d, y_1^{a_1} \cdots y_r^{a_r})$. The map (5.2) factors as

$$\mathbb{A}_k^r \rightarrow \mathbb{A}_k^r = \text{Spec } k[\underline{z}] \rightarrow \mathbb{A}_k^{r+1}$$

where the first map is given by $(y_1, \dots, y_r) \mapsto (y_1^{a_1}, \dots, y_r^{a_r})$ and the second map is given by $(z_1, \dots, z_r) \mapsto (z_1^{d/a_1}, \dots, z_r^{d/a_r}, z_1 \cdots z_r)$. In short we have

$$\text{Spec } k[\underline{y}] \rightarrow \text{Spec } k[\underline{z}] \rightarrow X_{B_d} = \text{Spec} \left(k[\underline{x}, t] / (t^d - x_1^{a_1} \cdots x_r^{a_r}) \right)$$

and the associated morphisms of rings are the inclusions:

$$(5.3) \quad k[y_1, \dots, y_r] \supseteq k[y_1^{a_1}, \dots, y_r^{a_r}] \supseteq k[y_1^d, \dots, y_r^d, y_1^{a_1} \cdots y_r^{a_r}].$$

Let us consider for a moment the special case where $\text{Spec } k[\underline{z}] \rightarrow X_{B_d}$ is birational. This will be the case, for instance, if either $r = 2$, or, more generally, if $a_3 = \cdots = a_r = a_1 a_2$ (but will fail in general; e.g. the case $X = V(t - x_1^2 x_2 x_3)$). Then it follows from Zariski's main theorem that

$$\text{Spec } k[\underline{z}] \rightarrow X_{B_d}$$

is the normalization $\nu_d : X_d \rightarrow X_{B_d}$. The divisor D corresponds to (t) in $k[\underline{x}, t] / (t^d - x_1^{a_1} \cdots x_r^{a_r})$, which corresponds to $z_1 \cdots z_r$ in $k[\underline{z}]$. In conclusion, in this special case, X_d is smooth and $\pi_d^{-1}(0)$ is a reduced, nc divisor.

In general, describing the normalization $\nu_d : X_d \rightarrow X_{B_d}$ is more complicated. From the ring on the right in (5.3), one readily obtains a toric description of X_{B_d} . The normalization can then be described in terms of associated semi-groups (see [71, p.101]). Using this approach, it is established in [71, Lem. 1, p.102, Lem. 2, p.103] that $\pi_d^{-1}(0)$ is reduced, and the pair X_d and $\pi^{-1}(0)$ give rise to a toroidal embedding without self-intersection. We discuss toroidal embeddings briefly in the next section. In the case where X_d is non-singular, we point out that this implies that $\pi_d^{-1}(0)$ is nc.

Example 5.3. Again consider the variety X in $\text{Spec } k[\underline{x}, t]$ defined by $t - x_1^{a_1} \cdots x_r^{a_r}$. Using the same notation as in the previous example, we will keep the assumption that $d = \text{lcm}(a_1, \dots, a_r)$, but will discard the assumption that $\gcd(d, a_1, \dots, a_r) = \gcd(a_1, \dots, a_r) = 1$. Again, it is easy to describe $X_{B_d} := B_d \times_B X$; it is defined by $t^d - x_1^{a_1} \cdots x_r^{a_r}$. Setting $e = \gcd(d, a_1, \dots, a_r)$, this family decomposes as $\prod_{\zeta^e=1} \left(t^{d/e} - \zeta \prod_{i=1}^r x_i^{a_i/e} \cdots x_r^{a_r/e} \right)$. Since $d/e = \text{lcm}(a_1/e, \dots, a_r/e)$, and $\gcd(d/e, a_1/e, \dots, a_r/e) = 1$, we see that we can reduce to the case of (e copies of) the previous example.

Example 5.4. Again consider the variety X in $\operatorname{Spec} k[\underline{x}, t]$ defined by $t - x_1^{a_1} \cdots x_r^{a_r}$. Using the same notation as in the previous example, set $\ell = \operatorname{lcm}(a_1, \dots, a_r)$, assume that $d = n \cdot \ell$ for some $n \in \mathbb{N}$, and again discard the assumption that $\gcd(d, a_1, \dots, a_r) = \gcd(a_1, \dots, a_r) = 1$. One can show (see e.g. [71, Lemma 2, p.103]) that $X_d = B_d \times_{B_\ell} X_\ell$.

Remark 5.5. In summary, for the variety $X \subseteq \operatorname{Spec} k[\underline{x}, t]$ defined by $t - x_1^{a_1} \cdots x_r^{a_r}$, in the notation above if $d = n \cdot \operatorname{lcm}(a_1, \dots, a_r)$ for some $n \in \mathbb{N}$, then X_d consists of $e = \gcd(a_1, \dots, a_r)$ connected components. On each of these components, $\pi_d^{-1}(0)$ gives rise to a toroidal embedding without self-intersection. For surfaces, the singularities appearing on X_d will be at worst of type A (the definition of a type A singularity is recalled in §6).

We now briefly outline the Mumford et al. proof of the Semi-stable Reduction Theorem (see [71, pp.98-108] for more details).

Sketch of the proof of the Semi-stable Reduction Theorem. Let $\pi : X \rightarrow B$ be a morphism as in the statement of the theorem. Using the characteristic zero assumption, perform a log resolution of the pair $(X, \pi^{-1}(o))$. We obtain a new family $\tilde{\pi} : \tilde{X} \rightarrow B$, where $\tilde{\pi}^{-1}(o)$ is normal crossing (although it may not be reduced). Setting ℓ to be the lcm of the multiplicities of the components of $\tilde{\pi}^{-1}(o)$, one makes a base change of degree ℓ , and then normalizes the total space.

Call the space obtained $X^\nu \rightarrow B_\ell$. The claim is that X^ν satisfies the conditions of the theorem, with the possible exception that X^ν may fail to be smooth (in which case the central fiber may only induce a toroidal embedding without self intersection, rather than being nc). Indeed, the question is étale local, so to describe X^ν we may reduce to the examples we have already considered, where \tilde{X} is defined by

$$t - \prod_{i=1}^r x_i^{a_i} \cdots x_r^{a_r}.$$

The remaining issue is to resolve the singularities of the total space of X^ν (while retaining the property that the central fiber induces a toroidal embedding without self-intersection). This is done in [71, pp.104-108]. \square

Remark 5.6. For the case of families of curves (where X is a surface), the total space of X^ν has type A singularities, and one can achieve the resolution in the final step above by a sequence of blow-ups introducing chains of rational curves.

To motivate some of the other examples considered in this survey, it is instructive to sketch a proof of a special case of the stable reduction theorem for curves in characteristic 0, using the semi-stable reduction theorem. The goal is to emphasize the role of semi-stable reduction and relative canonical models.

Sketch of stable reduction for curves in characteristic 0. For simplicity we consider the case of a smooth family of curves

$$\pi_K : X_K \rightarrow \operatorname{Spec} K$$

(of genus g) over the generic point of $S = \operatorname{Spec} R$, the spectrum of a DVR. Complete this to a family of schemes $\pi : X \rightarrow S$. Applying the semi-stable reduction theorem, one obtains after a finite base change a family of nodal curves $\pi' : X' \rightarrow S'$. The relative canonical model

$$\underline{\operatorname{Proj}}_{S'} \bigoplus_n \pi'_* \left(\omega_{X'/S'}^{\otimes n} \right) \rightarrow S'$$

is a family of stable curves extending the pull back of X_K . Let us denote this by $\pi^c : X^c \rightarrow S'$. Note that the relative canonical model of X^c/S' is again X^c/S' .

We now show the central fiber is determined up to isomorphism. To do this, suppose there were two stable reductions $\pi_1^c : X_1^c \rightarrow S'_1$ and $\pi_2^c : X_2^c \rightarrow S'_2$. Pulling back by a further finite base change, we may assume both stable reductions lie over the same base $S'_1 = S'_2 = S'$. The surfaces X_1^c and X_2^c are birational by construction. The claim is they are in fact isomorphic over S' . We outline the following standard proof of this in order to motivate similar statements in other settings.

Resolving the singularities of the surfaces, resolving the resulting birational map of smooth surfaces, and applying the Semi-stable Reduction Theorem again if necessary, we may assume there is a diagram:

$$\begin{array}{ccc} & Z & \\ \phi_1 \swarrow & & \searrow \phi_2 \\ X_1^c & & X_2^c \\ \pi_1^c \searrow & & \swarrow \pi_2^c \\ & S' & \end{array}$$

where ϕ_1, ϕ_2 are sequences of blow-ups, Z is a smooth surface, and Z/S' is a family of nodal curves. One can show that $(\pi_1^c)_* \omega_{X_1^c/S'}^{\otimes n} \cong (\pi_1^c \circ \phi_1)_* \omega_{Z/S'}^{\otimes n}$ (see e.g. [62, Ex. 3.108, p.156, p.84]) and similarly for the other side of the diagram. It follows that

$$(\pi_1^c)_* \omega_{X_1^c/S'}^{\otimes n} \cong (\pi_1^c \circ \phi_1)_* \omega_{Z/S'}^{\otimes n} \cong (\pi_2^c \circ \phi_2)_* \omega_{Z/S'}^{\otimes n} \cong (\pi_2^c)_* \omega_{X_2^c/S'}^{\otimes n}.$$

Thus the relative canonical models of X_1^c/S' and X_2^c/S' agree, so in fact X_1^c and X_2^c are isomorphic over S' . \square

5.2. Simultaneous semi-stable reduction. The question of extending the Semi-stable Reduction Theorem to higher dimensional bases is of course very natural, and was asked already in the introduction of [71]. We discuss some recent progress due to de Jong [42] and Abramovich–Karu [3].

5.2.1. Preliminaries. The first issue to address is what is meant by semi-stable reduction for higher dimensional bases. We take the following modification of the assumptions in the statement of the Semi-stable Reduction Theorem as the starting point. *We set B to be an open subset of a non-singular variety, set $U \subseteq B$ to be a non-empty open subset and suppose that $\pi : X \rightarrow B$ is a surjective, projective morphism of a variety X onto B so that the restriction $\pi_U : X|_U \rightarrow U$ is smooth.*

Our goal is to find a diagram as in (5.1) with B' nonsingular, f an alteration, p a projective modification, so that all of the geometric fibers of π' satisfy some natural conditions. For instance, at the very least, we would like all of the geometric fibers of π' to be reduced. Moreover, we could hope that all of the fibers have singularities that look at worst like smooth components meeting “transversally”.

For instance, when $\dim(X) = \dim(B) + 1$, if one allows the total space X' to be singular, then it is a result of de Jong [42, Thm. 5.8] that such a semi-stable reduction exists. Moreover, de Jong shows that in this case if p is permitted to be an alteration, rather than a modification, then X' can be taken to be smooth. For cases where the fibers are of arbitrary dimension, one can find a diagram and an open subset of B' with complement codimension two, satisfying the conditions above on the fibers by using the Semi-stable Reduction Theorem (see [3, §5], Kawamata [70, Thm. 17]).

Nevertheless, it is not possible in general to obtain a “semi-stable reduction” where the fibers all have singularities that at worst look like smooth components meeting transversally. For instance, the two parameter family of surfaces defined by

$$(5.4) \quad (t_1 - x_1x_2, t_2 - x_3x_4)$$

precludes this (see Karu [68, Exa. 1.12, p.21]). Thus, in general, one needs a different definition of “semi-stable reduction” to get a reasonable result.

5.2.2. A result of Abramovich–Karu. In light of the presentation in [71], and the family (5.4) above (which has fibers with at worst toric singularities), it is natural to change the focus to toroidal structures. Using this language, we state a theorem of Abramovich–Karu, and then discuss some definitions after the theorem.

Theorem 5.7 (Abramovich–Karu [3, Thm. 0.3]). *Assume $\text{char}(k) = 0$ and $k = \bar{k}$. Let $X \rightarrow B$ be a surjective morphism of projective varieties over k , with geometrically integral generic fiber. There exists a diagram as in (5.1) with X' a projective variety, B' nonsingular, f a projective alteration, p a projective strict modification, π' a toroidal morphism, and all of the geometric fibers of π' equidimensional and reduced.*

A toroidal structure on a normal variety X is an open subset $U_X \subseteq X$, such that for each $x \in X$, there is a toric variety X_{σ_x} , a point $s \in X_{\sigma_x}$ and an isomorphism $\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{X_{\sigma_x},s}$ that maps the ideal of $X - U_X$ to the ideal of $X_{\sigma_x} - T_{\sigma_x}$ where T_{σ_x} is the torus of X_{σ_x} . In other words, it is a

variety together with an open set that étale locally looks like a toric variety together with its embedded torus. A toroidal morphism is defined in the obvious way (see e.g. [3, Def. 1.3, p.247]). We direct the reader to [3, p.45] for the definition of a strict modification; we note that in the case that $X \rightarrow B$ is flat, p will be a projective modification.

It is mentioned in [3, Rem. 1.1] that it may also be possible to address simultaneous semi-stable reduction using the language of log-structures, rather than toroidal morphisms. We also direct the reader to [1], which addresses the case of schemes over fields that are not algebraically closed. We conclude with the remark that roughly speaking, the theorem says that simultaneous semi-stable reduction is possible if one allows for toric singularities.

Remark 5.8. It would be interesting to know if by restricting the class of singularities in the fibers of $X \rightarrow B$, one could give further information on the structure of the morphisms f , p , and π' , and in particular, give further restrictions on the singularities of the fibers of π' . For instance, in the next section we will recall the definition of semi log canonical singularities in regards to results of Kollár on a proper moduli functor for canonically polarized manifolds. The question of simultaneous stable reduction could be asked in that context as well, and we will return to this question in the next section.

5.3. Stable reduction and proper moduli. The second approach we can take to the stable reduction problem for higher dimensional varieties is to look for a class of schemes so that the central fiber of a one-parameter family of such schemes can be filled in (after a finite base-change) uniquely (up to isomorphism) with a scheme in this class. This corresponds to the valuative criterion of properness for the associated moduli stack. For schemes of dimension one, the class of stable curves gives an example.

In general, determining a proper moduli stack, or even a separated moduli stack, is quite difficult. The literature on this topic is vast, and we direct the reader to Viehweg [102], Alexeev [6, 7], Kollár–Shepherd-Baron [78] and Kollár [73, 75, 76]. Here, we will discuss recent results of Kollár [75] that extend these results to give a stable reduction theorem for canonically polarized varieties.

5.3.1. Preliminaries. We begin by reviewing a few definitions. *All schemes will be taken to be reduced, of finite type over \mathbb{C} , and all points will be taken to be closed points, unless otherwise stated.* Recall a node of an equidimensional scheme X of dimension n is a point $x \in X$ such that $\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[x_1, \dots, x_{n+1}]]/(x_1x_2)$ as \mathbb{C} -algebras. X is said to have at worst nodes in codimension 1 if there exists an open subset $V \subseteq X$ with $\text{codim}_X(X - V) \geq 2$, with the property that for all $x \in V$, x is either a non-singular point, or a node. For scheme X that is S_2 , X is nodal in codimension one if

and only if, in codimension one, it is both semi-normal and Gorenstein (see Kollár–Kovács [79, §5.1]).

We will want to discuss divisors on reducible, equidimensional, reduced schemes X . A Weil divisor D on such a scheme is a finite, formal, integral, linear combination of (not necessarily closed) points $E \in X$ such that $\mathcal{O}_{X,E}$ is a DVR. There is a notion of linear equivalence for such divisors obtained via Weil divisorial subsheaves; we direct the reader to Corti [40, (16.1.1), (16.2.2), p.171-2]. A \mathbb{Q} -divisor is defined similarly, with \mathbb{Q} -coefficients. A \mathbb{Q} -divisor D on X is said to be \mathbb{Q} -Cartier if there exists an $m \in \mathbb{N}$ such that mD is the Weil divisor associated to a Cartier divisor.

For X a projective scheme, we will refer to the dualizing sheaf $\omega_{X/k}$ as the canonical sheaf. If X is Gorenstein in codimension one, then associated to $\omega_{X/k}$ is a linear equivalence class of Weil divisors (see e.g. [40, (16.3.3), p.173]). We denote this equivalence class by K_X and call it the canonical divisor (class). We direct the reader also to [77, §5.5] and Kollár–Kovács [79, p. 11] for more discussion.

Remark 5.9. In order to limit the length of this survey, we have suppressed the notion of a pair in most of the topics covered. However, the utility of parameterizing varieties together with a distinguished divisor goes back at least to the case of principally polarized abelian varieties, where the canonical bundle is trivial, and one substitutes the theta divisor in its place to provide a natural rigidity to the problem. Recently it has become clear that in many other situations it can be beneficial to consider pairs (X, Δ) where X is a variety, and Δ is an effective divisor so that $K_X + \Delta$ is ample (see especially the work of Kollár and Alexeev cited above). The notion of pairs will be central in what follows.

5.3.2. Semi log canonical models. We start by recalling the definition of log canonical pairs. Let X be a projective, reduced, equidimensional, S_2 scheme and let Δ be an effective \mathbb{Q} -divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. With these assumptions, we say that the pair (X, Δ) is **log canonical (lc)** if X is smooth in codimension one (or equivalently X is normal) and there exists a log resolution $f : Y \rightarrow X$ of (X, Δ) such that

$$(5.5) \quad K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i,$$

where the E_i are f -exceptional divisors and $a_i \geq -1$ for every i . Note that the equality in (5.5) is \mathbb{Q} -equivalence of \mathbb{Q} -Cartier divisors. We say that X is **lc** if the pair $(X, 0)$ is lc, where 0 is the zero divisor (see e.g. [77] for more discussion).

With the assumptions above (in italics), we say that the pair (X, Δ) is **semi log canonical (slc)** if X is nodal in codimension one (or equivalently, in codimension one X is seminormal and Gorenstein), $K_X + \Delta$ is \mathbb{Q} -Cartier, and if $\nu : X^\nu \rightarrow X$ is the normalization of X and Θ is the \mathbb{Q} -Weil divisor

on X given by

$$(5.6) \quad K_{X^\nu} + \Theta = \nu^*(K_X + \Delta),$$

then the pair (X^ν, Θ) is lc. Note that the equality in (5.6) is an equivalence, for which we refer the reader to Kollár–Kovács [79, (5.7.5), Def.-Lem. 5.10]. We say that X is **slc** if the pair $(X, 0)$ is slc, where 0 is the zero divisor (see e.g. Abramovich–Fong–Kollár–McKernan [2], Fujino [54], Kollár [75] and Kollár–Kovács [79, §5.2] for more discussion).

A **semi log canonical model** (slc model) is an slc scheme X such that K_X is ample ([75, Def. 15]). In particular, if X is smooth, then it is an slc model if and only if it is canonically polarized. A motivation for this definition also comes from the cases of curves and surfaces. A semi log canonical model of dimension one is a stable curve of genus $g \geq 2$. A result of Kollár–Shepherd-Barron [78, Cor. 5.7], Kollár [73, Cor. 5.6] and Alexeev [5] establishes that there is a proper moduli space of semi log canonical models of dimension two (with fixed invariants K_X^2 and $\chi(\mathcal{O}_X)$). The valuative criterion for properness of the moduli space can be established with an appropriate stable reduction theorem.

5.3.3. Kollár’s stable reduction theorem. Recently Kollár has established a stable reduction theorem for semi log canonical models of any dimension. The full statement would require introducing the notion of relative semi log canonical models (and in particular the notion of reflexive hulls on non-normal schemes), which we omit (see [75, Def. 28, 29]). Below we state a weaker version of this stable reduction theorem, where the generic fiber of the family is lc. We sketch the parts of the proof that are formally similar to the proof of the stable reduction theorem that we sketched in the case of curves.

Theorem 5.10 (Kollár [75, 5.38]). *Let B be an open subset of a non-singular curve over \mathbb{C} , fix a point $o \in B$, and set $U = B - \{o\}$. Suppose that*

$$\pi : X \rightarrow B$$

is a flat, projective, morphism with connected fibers, such that the restriction $\pi_U : X_U = X|_U \rightarrow U$ has lc fibers, and has π_U -ample relative dualizing sheaf $\omega_{X_U/U}$. Then there is a finite base change $f : B' \rightarrow B$, with B' non-singular and $f^{-1}(o)$ a single point o' , and a scheme

$$\pi^c : X^c \rightarrow B'$$

such that X^c and $B' \times_B X$ are isomorphic over $U' := B' - o'$, and the fiber $X_{o'}^c = (\pi^c)^{-1}(o')$ is an slc model. Moreover, the fiber $X_{o'}^c$ is unique up to isomorphism.

Sketch of the proof. Let $\pi : X \rightarrow B$ be as in the statement of Theorem 5.10. From the Semi-stable Reduction Theorem we obtain a diagram as in (5.1) including a morphism $\pi' : X' \rightarrow B'$ satisfying the conclusions of Theorem 5.10, except that the central fiber of the morphism $\pi' : X' \rightarrow B'$, which is

normal crossing, may not have ample canonical class (see Hacon–Xu [61] for the case where the general fiber is lc, rather than smooth). From this point, motivated in part by the case of curves, one considers

$$X^c := \underline{\mathrm{Proj}}_{B'} \left(\bigoplus_{k=0}^{\infty} \pi'_* \left(\omega_{X'/B'}^{\otimes k} \right) \right).$$

A result of Birkar–Cascini–Hacon–McKernan [24, Thm. 1.2 (3)] implies that the sheaf $\bigoplus_{k=0}^{\infty} \pi'_* (\omega_{X'/B'}^{\otimes k})$ is finitely generated as an $\mathcal{O}_{B'}$ -algebra (see also Hacon–Xu [61] for the case where the general fiber is lc). In other words the projection $\pi^c : X^c \rightarrow B'$ is a projective morphism that agrees with $\pi' : X' \rightarrow B'$ over U' . One can show that the central fiber of π^c is an slc model ([75]).

It remains to show that the central fiber is determined up to isomorphism. One does this by establishing that any other projective morphism $\hat{\pi}^c : \hat{X}^c \rightarrow B'$ with $K_{\hat{X}^c}$ a \mathbb{Q} -Cartier divisor, which agrees with $\pi^c : X^c \rightarrow B'$ over U' , and which has central fiber an slc model, is isomorphic to $\pi^c : X^c \rightarrow B'$ over B' . The proof in the case where the generic fiber is smooth is formally similar to the proof we sketched in the case of curves. We direct the reader to Kollár [Pro. 6, Def. 7, Def. 15, and pp.8-9] for more details (see also [23, Lem. 2.7]). \square

Remark 5.11. It is well known that a smooth, projective variety of general type has a finite automorphism group (positive dimensional automorphism groups give rise to rational curves or abelian varieties covering the variety). More generally, it is a result of Iitaka that smooth, projective varieties of log general type have finite automorphism groups [66, Lem. 1, p.87, Def. p.71]. Consequently, considering log resolutions of each irreducible component of the normalization, one would expect from (5.5) and Iitaka’s result that an slc model would have a finite automorphism group; this is in fact the case [78, p.328], [79, Cor. 9.71], [23, Lem. 2.5].

Example 5.12. It is interesting to note the importance of having a condition such as slc in the remark above. For instance, a plane quartic C consisting of two smooth conics meeting in a single point, which is a tacnode, has ample canonical bundle $\mathcal{O}_C(1)$. However, the automorphism group of the curve is not finite (see §7.3 below). Gluing copies of \mathbb{P}^2 along such curves, one can construct similar examples of surfaces.

Remark 5.13. More generally, a simultaneous stable reduction theorem holds for slc models. For dimension 1, this is de Jong’s theorem. For dimension 2, this is a result of Alexeev, Kollár and Kollár–Shepherd-Barron. For higher dimensions, one may use Kollár’s stable reduction theorem together with the results of Edidin et al. and Fedorchuk discussed in §4 (i.e. Theorem 4.10 and Lemma 4.8).

More precisely, fix a function $H : \mathbb{Z} \rightarrow \mathbb{Z}$, and let $\overline{\mathcal{M}}_H$ be the associated category fibered in groupoids (over the category $\mathrm{Sch}_{\mathbb{C}}$ of schemes over \mathbb{C}),

with objects that are families of slc models as defined in [75, Def. 29], and with morphisms given by pull-back diagrams. It is shown in [23, Thm. 2.8] (using recently announced results of Hacon–McKernan–Xu) that $\overline{\mathcal{M}}_H$ is a proper Deligne–Mumford stack (over the étale site $\text{Sch}_{\mathbb{C}}^{\text{ét}}$). Consequently, given a quasi-compact, quasi-separated (or Noetherian) scheme B (over \mathbb{C}) and a rational map $B \dashrightarrow \overline{\mathcal{M}}_H$, there exists an alteration $\tilde{B} \rightarrow B$ resolving the rational map.

6. (SEMI-)STABLE REDUCTION FOR SINGULARITIES

We now consider the stable reduction problem locally and focus on singularities. Since we are somewhat farther from a moduli problem, we consider initially the question of semi-stable reduction; i.e. locally analytically replacing the central fiber of a family with something “nicer”. Typically, nicer will mean something in normal crossings position. We direct the reader to §1, which can be interpreted as a (simultaneous) semi-stable reduction of a deformation of a cusp singularity. For the remainder of this section, we will work over \mathbb{C} .

The Mumford et al. Semi-stable Reduction Theorem for one-parameter families ensures the existence of a semi-stable reduction for (generically smooth) one-parameter families of singularities. The extensions to higher dimensional bases due to Abramovich–Karu establish a certain form of existence in the simultaneous case. Consequently, the problem we will consider here is describing in more detail the central fiber of semi-stable reductions for specific singularities.

Singularities of type *ADE* will appear frequently in what follows. Recall that these are the singularities (of dimension $n - 1$, $n \geq 2$) defined by the polynomials:

$$\begin{aligned} f_{A_k} &= x_1^{k+1} + x_2^2 + \dots + x_n^k & k \geq 1 \\ f_{D_k} &= x_1(x_1^{k-2} + x_2^2) + x_3^2 + \dots + x_n^2 & k \geq 4 \\ f_{E_6} &= x_1^4 + x_2^3 + x_3^2 + \dots + x_n^2 \\ f_{E_7} &= x_2(x_1^3 + x_2^2) + x_3^2 + \dots + x_n^2 \\ f_{E_8} &= x_1^5 + x_2^3 + x_3^2 + \dots + x_n^2. \end{aligned}$$

6.1. Local stable reduction for curve singularities. In this section we discuss some recent work of Hassett [63] on semi-stable reduction for isolated, locally planar singularities. The main results are descriptions of the tails arising in the stable reduction process for curves.

6.1.1. Preliminaries on local stable reduction. A local stable reduction of an isolated, plane curve singularity (X_o, x) is defined as follows. We consider

$$\pi : X \rightarrow B$$

a one-parameter smoothing of (X_o, x) , with $X_o = \pi^{-1}(o)$ for some $o \in B$; one can obtain such a smoothing by observing that the singularity (X_o, x) will arise on some plane curve, and the Hilbert scheme containing that curve is a

projective space with generic point parameterizing a smooth curve. We then perform semi-stable reduction following Mumford et al. to obtain $\tilde{X} \rightarrow \tilde{B}$, where the central fiber is in nc position. Set

$$p : \tilde{X} \rightarrow \tilde{B} \times_B X.$$

Finally, take the log canonical model of (\tilde{X}, \tilde{X}_o) relative to the morphism p .

We will denote the resulting family by

$$X^c \rightarrow \tilde{B};$$

this is called the **local stable reduction** of the family $X \rightarrow B$. Note that X^c agrees with $\tilde{B} \times_B X$ away from the central fiber. By construction, the local stable reduction provides a local picture of the stable reduction for a one-parameter family of curves degenerating to a curve with a singularity (X_o, x) .

We now review the definition of the tail of the local stable reduction. The central fiber of $X^c \rightarrow B'$, which we will denote X_o^c , can be decomposed as $X_o^c = X_o^{(1)} \cup X_o^{(T)}$, where $X_o^{(1)}$ is the normalization of X_o and $X_o^{(T)} := \overline{X_o^c - X_o^{(1)}}$. To fix notation, set $X_o^{(1)} \cap X_o^{(T)} = \{p_1, \dots, p_b\}$ where b is the number of branches of X_o . The pair

$$(X_o^{(1)}, \{p_1, \dots, p_b\})$$

depends only on X_o and not on the choice of smoothing. On the other hand, the pair

$$(X_o^{(T)}, \{p_1, \dots, p_b\})$$

may depend on the smoothing, and we call this the **tail of the local stable reduction** of the family $X \rightarrow B$.

6.1.2. A result of Hassett. We now mention Hassett's result that the tails arising in this process form subvarieties of the moduli space of curves. We will use the notation $\overline{M}_{g,(n)} = \overline{M}_{g,n}/\Sigma_n$ for the moduli space of stable curves of genus g , with n *unordered* marked points.

Proposition 6.1 (Hassett [63, Prop. 3.2, p.176]). *Let (X_o, x) be a plane curve singularity with b branches. Let \mathcal{T}_{X_o} be the set of tails obtained from the local stable reduction of each smoothing of X_o . The tails are connected, all of the same arithmetic genus γ , and \mathcal{T}_{X_o} is naturally a (reduced) subscheme of $\overline{M}_{\gamma,(b)}$.*

In order to describe the subvariety \mathcal{T}_{X_o} in more detail, Hassett considers the problem of deforming the pairs $(\text{Spec } \mathbb{C}[[x, y]], X_o)$. He considers a process similar to that in the construction of the local stable reduction, performing semi-stable reduction for the pair $(\text{Spec } \mathbb{C}[[x, y]], X_o)$ and then taking a log-canonical model. His results give explicit descriptions of tails that arise in stable reduction for a wide class of singularities, including the classes known as toric and quasi-toric singularities (which include *ADE*

singularities). For the sake of space, we restrict to the special case of A_n singularities.

Theorem 6.2 (Hassett [63, Thm. 6.2,6.3, p.185-6]). *Suppose that (X_o, x) is a plane curve singularity of type A_n . Then the scheme \mathcal{T}_{X_o} is irreducible.*

- (1) *If $n = 2k$, then \mathcal{T}_{X_o} is the closure of the locus of hyperelliptic curves of genus k , with a marked Weierstrass point in $\overline{M}_{k,1}$.*
- (2) *If $n = 2k + 1$, then \mathcal{T}_{X_o} is the closure of the locus of hyperelliptic curves of genus k with two conjugate marked points (i.e. interchanged by the hyperelliptic involution) in $\overline{M}_{k,(2)}$.*

Remark 6.3. One application of these results is to the Hassett–Keel program. More precisely, the results can be used to provide a description of resolutions of rational maps among various moduli spaces that arise in the program. We direct the reader to [35, §4.2] for more discussion (see also §4.4 above).

6.2. Simultaneous stable reduction for simple surface singularities and the Weyl cover. We now turn our attention to surface singularities, again over \mathbb{C} . While in general one would consider questions of semi-stable reduction, for surface singularities of type ADE there is a result due to Brieskorn–Tyurina that one may in fact find simultaneous resolutions of singularities. We discuss the meaning of this result, as well as its peculiarity to dimension two. We then state the result precisely. An additional motivation is to introduce the so-called Weyl cover of the base of a mini-versal deformation of an ADE singularity, a topic that arises frequently, and was discussed before in the context of stable reduction for curves in §4.4.

First let us recall what is meant by a simultaneous resolution of singularities. Let $\pi : X \rightarrow B$ be a flat morphism of schemes. A simultaneous resolution of singularities of π is a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ \pi' \downarrow & & \downarrow \pi \\ B & \xlongequal{\quad} & B \end{array}$$

such that p is proper, π' is smooth, and for every $b \in B$, the induced morphism $X'_b \rightarrow X_b$ is birational; i.e. it is a coherent way of resolving the singularities of the fibers of π .

Let us now make the following observation ([77, Exa. 4.27, p.128]): If B is a curve, and π is smooth over $B - \{o\}$ for some $o \in B$, then π does not admit a simultaneous resolution if X_o is a reduced curve, or $\dim X_o \geq 3$ and X_o has at worst isolated hypersurface singularities. With this in mind, Brieskorn’s theorem on surface singularities becomes quite surprising.

Theorem 6.4 (Brieskorn–Tyurina). *Let $\pi : (X, x) \rightarrow (B, o)$ be a flat morphism of germs of singularities such that fiber (X_o, x) is an ADE surface singularity. Then there is a finite, surjective morphism $(B', o') \rightarrow (B, o)$*

such that $\pi' : X' := B' \times_B X$ admits a simultaneous resolution of singularities.

We direct the reader to Kollár–Mori [77, p.129] for a discussion of a number of techniques that can be used to prove the theorem, as well as some references. Brieskorn’s [29] Weyl group cover of the mini-versal deformation space of an *ADE* singularity plays an important role, and we discuss this in more detail now.

6.2.1. The Weyl cover. Let X_o be an *ADE* singularity of type T (i.e. $T = A_n, D_n$ or E_n). Let B_T be a mini-versal deformation space of X_o with discriminant Δ_T . Define W_T to be the Weyl group of type T and R_T be the corresponding root system. Brieskorn shows there exists a Galois cover $f : B'_T \rightarrow B_T$ with covering group W_T and ramification locus Δ_T such that $f^*\Delta_T$ is an arrangement of hyperplanes determined by the root system R_T . The hyperplanes are in one-to-one correspondence with the roots in R_T considered up to ± 1 . The morphism $f : B'_T \rightarrow B_T$ is called the **Weyl (group) cover**.

In concrete terms, the Weyl cover of type T is given by Chevalley’s Theorem, which says that the subalgebra of W_T -invariant polynomials is a polynomial ring itself; i.e. it is given by

$$B' = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n]) \rightarrow B = \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_n]^{W_T}).$$

A Weyl group is a finite reflection group; the hyperplane arrangement \mathcal{H} is simply the set of (complexified) reflection hyperplanes. It is clear that $\sum_{H \in \mathcal{H}} H$ is the ramification divisor of $f : B' \rightarrow B$, and that $f_* \sum_{H \in \mathcal{H}} H = (\sum_{H \in \mathcal{H}} H) / W$ is the branch divisor. Brieskorn’s theorem ([29]) asserts that this branch divisor is the discriminant in a mini-versal deformation space of a singularity of type T (see also [35, §2] for some further discussion and references).

Remark 6.5. For surfaces, *ADE* singularities are exactly the canonical singularities (see e.g. [77]). Thus this special case is enough to handle surface singularities in many circumstances. We direct the reader to [78] for a complete description of slc surface singularities.

7. GEOMETRIC INVARIANT THEORY

A basic result in Geometric Invariant Theory (GIT) gives some insight into the problem of stable reduction in the case of a moduli space obtained as a GIT quotient. We refer the reader to Mumford [88] for a few of the basic definitions and results on GIT.

7.1. Preliminaries on GIT. Let X be a projective variety over an algebraically closed field k . Let G be a linearly reductive algebraic group over k [88, Def. 1.4, p.26] acting on X , and let L be an ample G -linearized line bundle on X [88, Def. 1.6, p.30].

For $n \in \mathbb{N}$ and a section $s \in H^0(X, L^{\otimes n})$, we set

$$X_s = \{x \in X : s(x) \neq 0\}.$$

Recall from [88, Def. 1.7] that the set of semi-stable (resp. stable, resp. properly stable) points of X , denoted X^{ss} (resp. X^s , resp. X_0^s), is the set of points $x \in X$ such that there exists a natural number n and a G -invariant section $s \in H^0(X, L^{\otimes n})^G$ with $s(x) \neq 0$ (resp. $s(x) \neq 0$ and the action of G on X_s closed, resp. $s(x) \neq 0$, the action of G on X_s closed, and the dimension of the stabilizer of x is equal to 0). We denote the orbit of x by $G \cdot x$ and the stabilizer of x by G_x [88, p.3].

Mumford's theorem [88, Theorem 1.10] defines the GIT quotient of X under the group action. It states that there exists a surjective G -invariant morphism of k -schemes

$$\phi : X^{ss} \rightarrow X//_L G$$

that is a categorical quotient of X^{ss} by the action of G [88, Def. 0.5, p.3]. This satisfies the additional property that if x_1 and x_2 are closed points of X^{ss} , then $\phi(x_1) = \phi(x_2)$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset$ ([88, p.40]). In particular, the closed points of $X//_L G$ are in bijection with closed orbits of closed points in X^{ss} . There is an open subset $(X//_L G)^\circ \subseteq X//_L G$ with the property that $X_0^s = \phi^{-1}(X//_L G)^\circ$ ([88, (1) p.37]), and the induced morphism

$$X_0^s \rightarrow (X//_L G)^\circ$$

is a geometric quotient; in particular the fibers over closed points are exactly the orbits of the closed points of X_0^s (see [88, Def. 0.6, p.4]). It is also shown that ϕ is universally open (see [88, (4) p.6]) and that

$$X//_L G = \text{Proj} \left(\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})^G \right)$$

(see [88, p.40], [45, Prop. 8.1, p.120]) so that $X//_L G$ is projective.

A commonly used tool in the study of the local structure of the GIT quotient is the Luna Slice Theorem [82, p.79] (see also [20, Prop. 7.6]). We recall the main consequence of the theorem here. Let $x \in X^{ss}$ and assume that the orbit $G \cdot x$ is closed and the stabilizer G_x is reduced (this always holds in characteristic zero). Then there exists an affine subscheme $W \subseteq X$ containing x that is stabilized by G_x (i.e. $G_x \cdot W \subseteq W$) and an étale morphism

$$W/G_x \rightarrow X//_L G.$$

The scheme W is called a Luna slice through x , and in words the theorem states that for a semi-stable point $x \in X^{ss}$ with closed orbit, the GIT quotient in a neighborhood of $\phi(x)$ is étale locally isomorphic to the quotient of a Luna slice through x by the action of the stabilizer of x .

7.2. Stable reduction for GIT. In the situation above, $X//_L G$ is proper. In particular, any map from the generic point of a DVR (over k) to $X//_L G$ extends to the whole DVR. One can also consider the question of lifting such maps from $X//_L G$ to X^{ss} .

Remark 7.1. The motivation, which we will discuss further in an example below, is that in the construction of moduli spaces via GIT, it is often the case that a universal family exists over X , but not over $X//_L G$, and one wants to know the stronger statement of how a one-parameter family can be extended over a punctured disk (not just how the map of the base of the family extends to a map to the moduli space).

Let us now make this lifting question more precise. Let R be a DVR over k , with fraction field $K = K(R)$, and with residue field $\kappa(R) = k$. Let $S = \text{Spec } R$, let $\eta = \text{Spec } K$ be the generic point, and let $s = \text{Spec } \kappa(R)$ be the special point. We will assume we are given a map

$$f : S \rightarrow X//_L G$$

with the property that $f(\eta) \subseteq (X//_L G)^\circ$. We are interested in lifting f to X^{ss} . We start by considering the problem of lifting $f_\eta : \eta \rightarrow (X//_L G)^\circ$ to X_0^s . The first observation is that *there exists a DVR R' with field of fractions K' , a finite map $\text{Spec } R' \rightarrow \text{Spec } R$, and a commutative diagram*

$$(7.1) \quad \begin{array}{ccc} & & X_0^s \\ & \nearrow g_{\eta'} & \downarrow \phi|_{X_0^s} \\ \text{Spec } K' & \longrightarrow \text{Spec } K & \xrightarrow{f_\eta} (X//_L G)^\circ. \end{array}$$

This follows immediately from the Luna Slice Theorem and the fact that we have restricted to X_0^s , where by definition the stabilizers are finite.

The main result of this section is that an analogous statement holds for lifts of $f : S \rightarrow X//_L G$ to X^{ss} (Shah [98, Prop. 2.1, p.488], Mumford [86, Lem. 5.3, p.57]).

Theorem 7.2 (Stable reduction for GIT). *Let $f : S \rightarrow X//_L G$ be a morphism such that $f(\eta) \in (X//_L G)^\circ$. There exists a DVR R' , a finite map $S' = \text{Spec } R' \rightarrow \text{Spec } R$, and a commutative diagram*

$$(7.2) \quad \begin{array}{ccc} & & X^{ss} \\ & \nearrow g & \downarrow \phi \\ S' & \longrightarrow S & \xrightarrow{f} X//_L G. \end{array}$$

Moreover, g may be chosen so that $g(s')$ lies in a closed orbit, where s' is the closed point of S' .

In light of (7.1), what this says is that, possibly after a further finite base change and an adjustment by the action of G , we may extend g_η to S , and

in fact do this in such a way that s is sent to a closed orbit. The proof of the theorem in [98] is a direct consequence of the universal openness of ϕ and a lifting lemma of Mumford [88, Lem., p.14] that can be applied to an irreducible component of the fibered product $S \times_{X//_L G} X^{ss}$.

7.3. GIT stable reduction for plane curves. In this section we consider the example of plane quartic curves, worked out by Mumford [88, Ch.4 §2]. We also direct the reader to [98, p.489] for a detailed analysis of the case of plane sextic curves, which provides another interesting example.

For plane quartic curves, we consider the associated Hilbert scheme $X = \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$. There is a natural action of $\mathbb{P}GL(3)$ given by change of coordinates; as is typical, for the sake of simplicity, we consider the action of $G = SL(3)$ instead, via the isogeny $SL(3) \rightarrow \mathbb{P}GL(3)$. The Hilbert scheme, being a projective space, comes equipped with a polarization $L = \mathcal{O}(1)$ and a natural $SL(3)$ -linearization. We set \overline{M}_3^{GIT} to be the GIT quotient

$$\overline{M}_3^{GIT} := \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) //_{\mathcal{O}(1)} SL(3) = X //_L G$$

Using the Hilbert–Mumford numerical invariant, the following is worked out in [88, p.81–2] (see also [16, Lem. 1.4]). Let C be a plane quartic corresponding to a point $x \in \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) = X$.

- (1) $x \in X_0^s$ if and only if C is non-singular, or C has only nodes and cusps as singularities.
- (2) $x \in X^{ss} - X_0^s$ if and only if C is a double conic or C has a tacnode.
- (3) $x \in X^{ss} - X_0^s$ and has closed orbit if and only if C is a double conic or C is the union of two conics, at least one of which is non-singular, and the conics meet tangentially.

We interpret this as follows. \overline{M}_3^{GIT} can be viewed as a space parameterizing isomorphism classes of curves of type (1) and (3). In particular, it contains an open subset parameterizing isomorphism classes of non-hyperelliptic curves of genus 3, and in this way we can consider \overline{M}_3^{GIT} as a compactification of this space.

While there is not a universal family of curves over \overline{M}_3^{GIT} , there is a universal family over X , and the GIT stable reduction theorem implies the following. Given any one-parameter family of plane quartics over a punctured disk, with fibers of type (1) above, after a finite base change, the family can be filled in to a family over the complete disk, with central fiber of type (1) or (3). Moreover, the isomorphism class of such a central fiber is determined by the original family over the punctured disk.

7.4. Deligne–Mumford stable reduction revisited. Gieseker’s construction of the moduli space of stable curves as a GIT quotient of a Hilbert scheme provides another proof of the Deligne–Mumford stable reduction theorem [56, p.i].

Let $g \geq 2$ and $\nu \geq 10$. Set $\text{Hilb}_{g,\nu}$ to be the irreducible component of the Hilbert scheme containing the locus of ν -canonically embedded, genus g ,

non-singular curves. Let $H_{g,\nu} \subseteq \text{Hilb}_{g,\nu}$ be the locus of (Deligne–Mumford) stable curves. Set $N = (2\nu - 1)(g - 1) - 1$ to be the dimension of ν -canonical space. The group $SL(N + 1)$ acts on $\text{Hilb}_{g,\nu}$ by change of basis. Gieseker has shown ([56, Ch.2]) that there exists an $SL(N + 1)$ -linearized polarization Λ on $\text{Hilb}_{g,\nu}$ such that

$$(7.3) \quad H_{g,\nu} = (\text{Hilb}_{g,\nu})_0^s = \text{Hilb}_{g,\nu}^{ss}.$$

Consequently, one obtains $\text{Hilb}_{g,\nu} //_{\Lambda} SL(N + 1) \cong \overline{M}_g$ [56, Thm. 2.0.2]. A key point is the fact that a family $X \rightarrow B$ of (Deligne–Mumford) stable curves over an affine scheme B can be embedded in \mathbb{P}_B^N as a flat family parameterized by a morphism $B \rightarrow \text{Hilb}_{g,\nu}$ (e.g. [56, p.13]). Since by definition \overline{M}_g is a coarse moduli space for (Deligne–Mumford) stable curves, the Deligne–Mumford stable reduction theorem follows from (7.3) and the GIT stable reduction theorem.

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